Modeling the Chaotic Waterwheel

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Outline

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   - Lorenz Equations
   - Waterwheel
2. Modeling the Waterwheel
   - Conservation of Mass
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   - Fourier Analysis
   - Equivalence
3. Simulation
   - Simulation

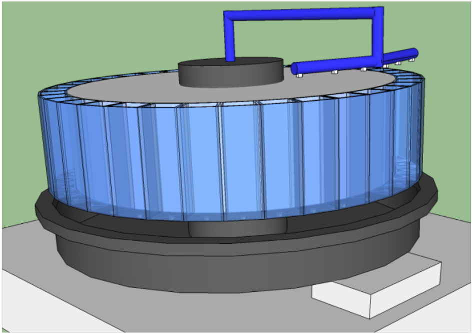
Definition

- Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Lorenz Equations

$$
\begin{aligned}
x' &= \sigma(y - x) \\
y' &= rx - y - xz \\
z' &= xy - bz
\end{aligned}
$$

Waterwheel



Variables

- θ = angle of wheel
- $\omega(t)$ = angular velocity of wheel
- $m(\theta, t)$ = mass distribution of water
- $Q(\theta)$ = inflow of water
- r = radius of wheel
- K = leakage rate
- v = rotational damping rate
- I = moment of inertia of the wheel

Change in Mass With Regard to Time

- Water pumped in by the nozzles ($Q(\theta)$)
- Water that leaks out ($Km(\theta, t)$)
- Rotation of the wheel ($\omega \frac{\partial m}{\partial \theta}$)

Equation

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$

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Torque Balance

- $I\omega'$ = damping torque + gravitational torque
- damping torque = $-v\omega$
- gravitational torque = $gr\int_0^{2\pi} m(\theta, t)\sin\theta\, d\theta$

Equation

$$I\omega' = -v\omega + gr\int_0^{2\pi} m(\theta, t)\sin\theta\, d\theta$$

Fourier Analysis

Fourier Series

$$m(\theta, t) = \sum_{n=0}^{\infty} [a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos(n\theta)$$

Fourier Analysis $m(\theta, t)$

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right] =$$

$$\sum_{n=0}^{\infty} q_n \cos n\theta$$

$$-K\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right]$$

$$-\omega\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right]$$

Fourier Analysis $m(\theta, t)$ II

$$a_n' = n\omega b_n - Ka_n$$
$$b_n' = -n\omega a_n - Kb_n + q_n$$

Fourier Analysis $\omega(t)$

$$I\omega' = -v\omega + gr \int_0^{2\pi} m(\theta, t) \sin\theta \, d\theta$$

$$I\omega' = -v\omega + gr \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] \sin\theta \, d\theta$$

$$I\omega' = -v\omega + gr \int_0^{2\pi} a_1 \sin^2\theta \, d\theta$$

$$I\omega' = -v\omega + \pi g r a_1$$

Final Equations

$$a_n' = n\omega b_n - Ka_n$$
$$b_n' = -n\omega a_n - Kb_n + q_n$$
$$\omega' = (-v\omega + \pi g r a_1)/I$$

Final equations

$$a_1' = n\omega b_1 - Ka_1$$
$$b_1' = -n\omega a_1 - Kb_1 + q_1$$
$$\omega' = (-v\omega + \pi g r a_1)/I$$

Change of Variables

- $a_1 = \alpha y$
- $b_1 = \beta z + q_1/K$
- $\omega = \gamma x$
- $t = T\tau$

$$\frac{\partial \alpha y}{\partial T\tau} = [\gamma x][\beta z + q_1/K] - K[\alpha y]$$

$$\frac{\partial [\beta z + q_1/K]}{\partial T\tau} = -[\gamma x][\alpha y] - K[\beta z + q_1/K] + q_1$$

$$\frac{\partial \gamma x}{\partial T\tau} = (-v[\gamma x] + \alpha\pi g r[\alpha y])/I$$

Change of Variables II

$T = 1/K, \gamma = \pm K$. Pick $\gamma = K$

- $a_1 = [Kv/\pi gr]\, y$
- $b_1 = [-Kv/\pi gr]\, z + q_1/K$
- $\omega = [K]\, x$
- $t = [1/K]\, \tau$
- $\sigma = v/KI$
- $r = \pi g r q_1 / K^2 v$

Simulation

Summary

- The Lorenz Waterwheel is a mechanical representation of the Lorenz equations.