A Ramsey bound on Jordan stable sets*

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Abstract

Jordan [2006] defined 'pillage games', a class of cooperative games whose dominance operator represents a 'power function' constrained by monotonicity axioms. In this environment, he proved that stable sets must be finite. We bound their cardinality above by a Ramsey number and show this bound to be tight for two agents. More generally, it is not tight as it does not make use of structural information across partial orders on the stable set.

Key words: pillage, cooperative game theory, stable sets

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Jordan [2006] introduced 'pillage games', a subset of abstract games in which the dominance relation is derived from a contest of power between competing coalitions. More specifically, a new allocation dominates an original one if the coalition that strictly prefers the new allocation is strictly more powerful, at the original allocation, than the coalition that strictly prefers the original. Power itself depends monotonically on coalitional membership, in the sense of set inclusion, and resource holdings.

Theorem 2.9 of Jordan [2006] proved that stable sets in pillage games are finite. The stark contrast between this result and those on stable sets in other cooperative games (q.v. most famously, Shapley [1959]) suggest that the monotonicity of domination in pillage games may allow much sharper results more generally.

The primary contribution of this note is to tighten the bound on the size of stable sets in Jordan [2006] pillage games. We show that this bound is tight for two agents. For three or more, the bound fails to use structural information across partial orders on the stable set, so will not generally be tight.

Let $I = \{1, \ldots, n\}$ be a finite set of *agents*. An *allocation* divides a unit resource among them, so that the feasible set is a compact, continuous $n-1$ dimensional simplex:

$$X \equiv \left\{ \{x_i\}_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}.$$

Following Jordan [2006], a *power function* is defined over subsets of agents and allocations, so that $\pi : 2^I \times X \to \mathbb{R}$ satisfies:

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(WC) if $C \subset C' \subseteq I$ then $\pi(C', \boldsymbol{x}) \geq \pi(C, \boldsymbol{x}) \, \forall \boldsymbol{x} \in X$;

(WR) if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

(SR) if $C \neq \emptyset \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Axiom WC requires weak monotonicity in coalitional inclusion; WR requires weak monotonicity in resources; SR requires strong monotonicity in resources.

An allocation $\boldsymbol{y}$ dominates an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$ iff

$$\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x});$$

where $W \equiv \{i \,|\, y_i > x_i\}$ and $L \equiv \{i \,|\, x_i > y_i\}$. By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric.

Lemma 1. *Axioms WC, WR and SR are independent.*

Proof. The 'weakest link' function, $\pi(C, \boldsymbol{x}) = \min_{i \in C} x_i$, satisfies axioms WR and SR but not WC. The constant function, $\pi(C, \boldsymbol{x}) = 1$, satisfies WC and WR but not SR. When $\delta(\cdot)$ is Kronecker's delta, axiom WR, but not the others, is violated by

$$\pi(C, \boldsymbol{x}) = v\|C\| + \min_{i \in C} x_i - \delta\left(\min_{i \in C} x_i\right)\left(\sum_{i \in C} x_i - \min_{i \in C} x_i\right);$$

for large v. The first term ensures satisfaction of WC. When x_i strictly increases for all $i \in C$, the first term is unchanged, the second strictly increases, and the third weakly increases. The third term penalises increased resources by members of C if some member still has nothing. □

Transposing an 'O-ring' production function [Kremer, 1993] to a pillage game yields $\pi(C, \boldsymbol{x}) = \prod_{i \in C} x_i$, which, like the weakest link function, also violates only WC. The constant function corresponds to strong property rights. Functions violating only WR are less compelling; in the example used in the proof, $\min_{i \in C} x_i$ can be replaced by $\prod_{i \in C} x_i$ or any other increasing function that remains constant when its least argument is zero.

Lemma 2. *Let W and L be non-empty subsets of I such that $W \cap L = \emptyset$. There are $2c \equiv 3^n - 2^{n+1} + 1$ such subsets.*

Proof. We can freely choose i elements for W, with $1 \leq i \leq n-1$. There are $\binom{n}{i}$ possibilities to do so for each i. For each of those, there are at most $n - i$ elements left for L. The full powerset of these, excepting the empty set, contains $2^{n-i} - 1$ elements. In total, then, there are $\sum_{i=1}^{n-1} \binom{n}{i} \cdot (2^{n-i} - 1)$ elements. Compute this using the binomial formula:

$$\begin{aligned}
\sum_{i=1}^{n-1} \binom{n}{i} \cdot \left(2^{n-i} - 1\right) &= \sum_{i=1}^{n-1} \binom{n}{i} \cdot 2^{n-i} - \sum_{i=1}^{n-1} \binom{n}{i} \cdot 1 \\
&= (3^n - 1 - 2^n) - (2^n - 1 - 1) \\
&= 3^n - 2^{n+1} + 1
\end{aligned}$$

□

For $n = 2$, $2c = 2$; for $n = 3$: $27 - 16 + 1 = 12$; for $n = 4$: $81 - 32 + 1 = 50$. We shall see in the Theorem why it is convenient to work with half this quantity.

A set of allocations, $S \subseteq X$, is a *stable set* iff it satisfies *internal stability*,

$$S \cap D(S) = \emptyset; \text{ and} \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

The conditions combine to yield $S = X \backslash D(S)$.[1]

This note's main result is that:

Theorem 1. *An internally stable set can contain at most $R_c(4) - 1$ allocations, the diagonal multicolour Ramsey number.*

Proof. Let S be an internally stable set. Any distinct $\boldsymbol{x}^i, \boldsymbol{x}^j \in S$ induce one of $2c$ distinct associated W and L sets. Form an undirected, c-coloured, complete graph as follows: let each $\boldsymbol{x}^i \in S$ be a node; for each $\boldsymbol{x}^i, \boldsymbol{x}^j \in S$, colour the edge between them according to the induced unordered pair, (W, L).

The proof of Theorem 2.9 [Jordan, 2006] establishes that there exists no sequence of allocations $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$ in S which holds W and L constant as the sequence progresses. As W and L induce a partial order on S, thus a transitive order, the existence of such a sequence would imply a monochromatic 4-clique in the graph.

The reverse is also true: a monochromatic 4-clique implies a sequence $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$ in S over which W and L are constant. To see why, note that the set of all possible sequences consistent with a monochromatic 4-clique may be identified with the set of paths on all complete, acyclic 4-graphs. (Maintaining constant W and L on a cyclic graph would require, by transitivity, $x_i > x_i$ for some agents i.) Examination of each such graph reveals at least one instance of such a sequence.

By definition of a Ramsey number, an undirected, c-coloured, complete graph with $R_c(4)$ nodes guarantees the existence of a monochromatic 4-clique, which cannot exist. The result follows. □

Figure 1 illustrates a 4-graph associated with a 4-clique: while there is a direct path from $\boldsymbol{x}^1$ to $\boldsymbol{x}^4$, the sequence $\left\{\boldsymbol{x}^1, \boldsymbol{x}^4\right\}$ may be augmented to include $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$ while maintaining constant W and L.

Example 1. *For $n = 2$, the theorem's upper bound is $R_1(4) - 1 = 3$, and therefore tight. This can be demonstrated with the 'wealth is power' function, $\pi_w(C, \boldsymbol{x}) = \sum_{i \in C} x_i$ [Jordan, 2006].*

For $n \geq 3$, the bound will generally not be tight as structural information across the partial orders is not used in the proof: $\boldsymbol{x} \multimap \boldsymbol{z}$ and $\boldsymbol{y} \multimap \boldsymbol{z}$ impose restrictions on the relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Example 2. *When $n = 3$, the upper bound is $R_6(4) - 1$, which is bounded by*

$$19{,}100{,}738 \geq R_6(4) > [R_3(4) - 1]^2 \geq 127^2 = 16{,}129.$$

The bounds are obtained through recursive application of inequality 5.a and direct calculation of inequality 5.2.n, respectively, in Radziszowski [2006].[2]

[1]The stable sets of combinatorial optimisation [Korte and Vygen, 2006] are game theory's internally stable sets: connect two nodes (allocations) with an edge if either of the two allocations dominates the other.

[2]The code for the former may be found at www.cs.bham.ac.uk/~mmk/demos/ramsey-upper-limit.lisp.

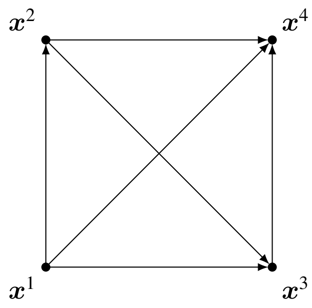

Figure 1: A 4-graph associated with a 4-clique: $\left\{x^i\right\}_{i=1}^4$ maintains constant W and L

For Jordan's wealth is power function, $\|S\| = 9$*; confining their attention to power functions satisfying an anonymity axiom, Kerber and Rowat [2008] have been unable to find a power function yielding* $\|S\| > 15$.

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