A Ramsey bound on stable sets in Jordan pillage games*

Manfred Kerber † Colin Rowat‡

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Abstract

Jordan [2006] defined 'pillage games', a class of cooperative games whose dominance operator is represented by a 'power function' satisfying coalitional and resource monotonicity axioms. In this environment, he proved that stable sets must be finite. We use graph theory to reinterpret this result, tightening the bound, highlighting the role played by resource monotonicity, and suggesting a strategy for yet tighter bounds.

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Jordan [2006] introduced 'pillage games', a class of cooperative games in which the dominance relation is derived from a contest of power between competing coalitions. More specifically, allocation $\boldsymbol{y}$ dominates allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, if the coalition that strictly prefers $\boldsymbol{y}$ is strictly more powerful, at $\boldsymbol{x}$, than the coalition that strictly prefers $\boldsymbol{x}$. Power itself depends monotonically on coalitional membership, in the sense of set inclusion, and resource holdings.

Resource monotonicity implies that no internally stable set can contain a sequence of four allocations, $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$, over which the coalition of agents strictly preferring $\boldsymbol{x}^{i+1}$ to $\boldsymbol{x}^i$ and that preferring the opposite are constant for $i = 1, 2, 3$.

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†School of Computer Science, University of Birmingham; m.kerber@cs.bham.ac.uk

‡Department of Economics, University of Birmingham; c.rowat@bham.ac.uk

By resource monotonicity, the power of the former coalition, W for 'win', would increase over the sequence, while the power of the latter, L for 'lose', would decrease. If such a sequence existed, one of its allocations would dominate another, violating internal stability.

Using this implication, Jordan [2006, Theorem 2.9] proved that stable sets in pillage games are finite. His result therefore contrasts starkly with results for other classes of cooperative games: Shapley's famous 'signature result' showed that uncountable stable sets could be built around arbitrary closed components [Shapley, 1959]. This note further exploits resource monotonicity to tighten Jordan's bound, replacing it with a Ramsey number. It does so by interpreting internally stable sets graph theoretically. This exploits transitive consequences of resource monotonicity unused in Jordan's proof: if $\left\{\boldsymbol{x}^i\right\}_{i=1}^2$ and $\left\{\boldsymbol{x}^i\right\}_{i=2}^4$ maintain constant coalitions W and L, then so does $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$.

With only two agents, any distinct allocations may be ordered to maintain constant coalitions, $W = \{1\}$ and $L = \{2\}$. Thus, for two agents, Example 1 shows that the upper bound presented here is tight. For three or more agents, this is no longer the case as there are sequences, $\left\{\boldsymbol{y}^i\right\}_{i=1}^4$ over which W grows in the sense of set inclusion and L shrinks in the same way; these violate internal stability but are not captured by methods that rely only on resource monotonicity. While it may be possible to derive tighter bounds yet from resource monotonicity alone, the huge gap between the Ramsey bound and the largest known examples of stable sets for three agents presented in Example 2 suggest that approaches that use coalitional monotonicity might be fruitful.

Thus, this note makes four contributions. First, by tightening the upper bound on the cardinality of their stable sets, it sharpens our knowledge of 'facts' pertaining to pillage games. Second, by interpreting the question graph theoretically, it continues a historical association between cooperative game theory and graph theory, deepening our understanding of the structure of pillage games' stable sets.[1] Third, by showing a tight bound only for two agents, it suggests that reliance on the resource monotonicity axiom alone may not suffice for deriving tight bounds; attempts to construct stable sets in particular applications will therefore need to use other techniques.[2] Finally, in doing so it also suggests a strategy for the derivation of yet tighter bounds.

[1] As Richardson [1953] observed, the stable set appears as a *Punktbasis zweiter Art* - a point basis of the second type - in the pioneering text on graph theory [König, 1936]. More recently, Brandt et al. [2007] proved that, when the dominance operator is irreflexive and asymmetric, the question of whether an allocation belongs to a stable set is $\mathcal{NP}$-complete in the number of possible allocations.

[2] See Jordan and Obadia [2004] for an algorithm for generating stable sets in pillage games when the core is non-empty.

Let $I = \{1, \ldots, n\}$ be a finite set of *agents*. An *allocation* divides a unit resource among them, so that the feasible set is a compact, continuous $n-1$ dimensional simplex:

$$X \equiv \left\{ \{x_i\}_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}.$$

Following Jordan [2006], a *power function* is defined over subsets of agents and allocations, so that $\pi : 2^I \times X \to \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$ then $\pi(C', \boldsymbol{x}) \geq \pi(C, \boldsymbol{x}) \, \forall \boldsymbol{x} \in X$;

(WR) if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

(SR) if $C \neq \emptyset \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Axiom WC requires weak monotonicity in coalitional inclusion; WR requires weak monotonicity in resources; SR requires strong monotonicity in resources.

An allocation $\boldsymbol{y}$ dominates an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \leftarrowtail \boldsymbol{x}$ iff

$$\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x});$$

where $W \equiv \{i \mid y_i > x_i\}$ and $L \equiv \{i \mid x_i > y_i\}$. By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric.

Axioms WC, WR and SR are independent. While perhaps unsurprising, this has not yet been formally established elsewhere, and the conditions required to violate only axiom WR may be of interest.

Lemma 1. *Axioms WC, WR and SR are independent.*

Proof. The 'weakest link' function, $\pi(C, \boldsymbol{x}) = \min_{i \in C} x_i$, satisfies axioms WR and SR but not WC. The constant function, $\pi(C, \boldsymbol{x}) = 1$, satisfies WC and WR but not SR. When $\delta(\cdot)$ is Kronecker's delta, axiom WR, but not the others, is violated by

$$\pi(C, \boldsymbol{x}) = v\|C\| + \min_{i \in C} x_i - \delta\left(\min_{i \in C} x_i\right)\left(\sum_{i \in C} x_i - \min_{i \in C} x_i\right);$$

for large v. The first term ensures satisfaction of WC. When x_i strictly increases for all $i \in C$, the first term is unchanged, the second strictly increases, and the third weakly increases. The third term penalises increased resources by members of C if some member still has nothing. □

Transposing an 'O-ring' production function [Kremer, 1993] to a pillage game yields $\pi(C, \boldsymbol{x}) = \prod_{i \in C} x_i$, which, like the weakest link function, also violates only WC. The constant function corresponds to strong property rights. Functions violating only WR are less compelling; in the example used in the proof, $\min_{i\in C} x_i$ can be replaced by $\prod_{i\in C} x_i$ or any other increasing function that remains constant when its least argument is zero.

We now turn to the paper's focus.

Lemma 2. *Let W and L be non-empty subsets of I such that $W \cap L = \emptyset$. There are $2c \equiv 3^n - 2^{n+1} + 1$ such subsets.*

Proof. We can freely choose i elements for W, with $1 \le i \le n-1$. There are $\binom{n}{i}$ possibilities to do so for each i. For each of those, there are at most $n-i$ elements left for L. The full powerset of these, excepting the empty set, contains $2^{n-i}-1$ elements. In total, then, there are $\sum_{i=1}^{n-1}\binom{n}{i}\cdot(2^{n-i}-1)$ elements. By the binomial formula:

$$\begin{aligned}\sum_{i=1}^{n-1}\binom{n}{i}\cdot\left(2^{n-i}-1\right) &= \sum_{i=1}^{n-1}\binom{n}{i}\cdot 2^{n-i} - \sum_{i=1}^{n-1}\binom{n}{i}\cdot 1\\ &= (3^n-1-2^n)-(2^n-1-1)\\ &= 3^n-2^{n+1}+1\end{aligned}$$

□

For $n=2$, $2c=2$; for $n=3$: $27-16+1=12$; for $n=4$: $81-32+1=50$. We shall see in the Theorem why it is convenient to work with half this quantity.

Let

$$D(Y) \equiv \{\boldsymbol{x} \in X \mid \exists \boldsymbol{y} \in Y \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\};$$

be the *dominion* of Y, the set of allocations dominated by an allocation in Y. Then a set of allocations, $S \subseteq X$, is a *stable set*[3] iff it satisfies *internal stability*,

$$S \cap D(S) = \emptyset; \text{ and} \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

This note's main result is that:

Theorem 1. *An internally stable set can contain at most $R_c(4)-1$ allocations, the diagonal multicolour Ramsey number.*

[3]The stable sets of combinatorial optimisation [Korte and Vygen, 2006] are game theory's internally stable sets: connect two nodes (allocations) with an edge if either of the two allocations dominates the other.

To aid understanding, we first define the graph theoretical terms used.[4] A *graph* (or *undirected graph*) $G = (V, E)$ with $E \subseteq [V]^2$ is a set of vertices, V, and a set of edges, E, connecting them. Vertices are *adjacent* if an edge connects them. A graph is *complete* if all pairs of its vertices are adjacent; it is *c-coloured* if each edge is represented by one of c colours. An *r-clique* is a complete subgraph of G on r vertices; it is *monochromatic* if all edges in the clique are of the same colour. A *directed graph* (or *digraph*) is a set of vertices and edges, (V, E), in which the edges are oriented from an initial vertex to a terminal vertex; a digraph is *acyclic* if it is impossible to return to any vertex along a path of its oriented edges. Ramsey's theorem guarantees that any sufficiently large, complete c-coloured graph will have a monochromatic r-clique; Ramsey's number, $R_c(r)$, is the least bound on the number of vertices in the graph required to make it "sufficiently large".

Proof. Let S be an internally stable set. By Lemma 2, any distinct $\boldsymbol{x}^i, \boldsymbol{x}^j \in S$ induce one of $2c$ distinct associated non-empty sets, W and L. Consequently, $\boldsymbol{x}^i$ and $\boldsymbol{x}^j$ induce one of c distinct unordered pairs, (W, L). Now form an undirected, c-coloured, complete graph as follows: let each $\boldsymbol{x}^i \in S$ be a vertex; for each $\boldsymbol{x}^i, \boldsymbol{x}^j \in S$, colour the edge between them according to the induced unordered pair, (W, L).

The proof of Theorem 2.9 [Jordan, 2006] established that there exists no sequence of allocations $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$ in S which holds W and L constant as the sequence progresses. Any such sequence induces a transitive relation on S as $i \leq j \leq k$ imply $i \leq k$. Thus, the subsequences $\left\{\boldsymbol{x}^1, \boldsymbol{x}^3\right\}$, $\left\{\boldsymbol{x}^1, \boldsymbol{x}^4\right\}$ and $\left\{\boldsymbol{x}^2, \boldsymbol{x}^4\right\}$ also maintain W and L constant, so that the existence of the sequence implies a monochromatic 4-clique in the graph.

The converse is also true: a monochromatic 4-clique implies a sequence $\left\{\boldsymbol{x}^i\right\}_{i=1}^4$ in S over which W and L are constant. To see why, note that the set of all possible sequences consistent with a monochromatic 4-clique may be identified with the set of paths on all complete, acyclic 4-digraphs. (Maintaining constant W and L on a cyclic graph would require, by transitivity, $x_i > x_i$ for some agent i.) Examination of each such graph reveals at least one instance of such a sequence.

By definition of a Ramsey number, an undirected, c-coloured, complete graph with $R_c(4)$ vertices guarantees the existence of a monochromatic 4-clique, which cannot exist. The result follows. □

Figure 1 illustrates a 4-digraph associated with a 4-clique: while there is a direct path from $\boldsymbol{x}^1$ to $\boldsymbol{x}^4$, the sequence $\left\{\boldsymbol{x}^1, \boldsymbol{x}^4\right\}$ may be augmented to include $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$ while maintaining constant W and L.

[4]See chapters 1 and 9 of Diestel [2000], from which the definitions here are taken, for more detailed treatment of the terms and theory.

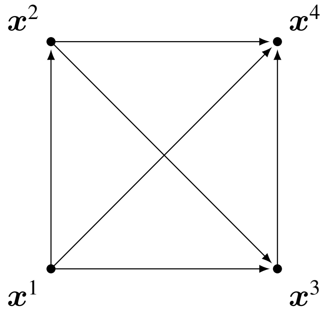

Figure 1: A 4-digraph associated with a sequence, $\{x^i\}_{i=1}^4$, that maintains constant W and L

Example 1. *For $n = 2$, the theorem's upper bound is $R_1(4) - 1 = 3$, and therefore tight. This can be demonstrated with the 'wealth is power' function, $\pi_w(C, x) = \sum_{i \in C} x_i$ [Jordan, 2006].*

Example 2. *When $n = 3$, the upper bound is $R_6(4) - 1$, which is bounded by*

$$16{,}129 = 127^2 \leq [R_3(4) - 1]^2 < R_6(4) \leq 19{,}100{,}738.$$

The bounds are obtained through direct calculation of inequality 5.n and recursive application of inequality 5.a, respectively, in Radziszowski [2006].[5] [6]

For Jordan's wealth is power function, $\|S\| = 9$. Confining their attention to power functions satisfying an anonymity axiom, Kerber and Rowat [2009] have been unable to find a power function yielding $\|S\| > 15$.

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[5] The code for the latter may be found at www.cs.bham.ac.uk/~mmk/demos/ramsey-upper-limit.lisp.

[6] Xiaodong et al. [2004, Theorem 2] constructed a graph satisfying the lower bound. The construction in Xiaodong [2002] added a further 595 vertices to the lower bound, for a new lower bound of 16,724.

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