Problem Solving

(Chapter 8 in Transitions)

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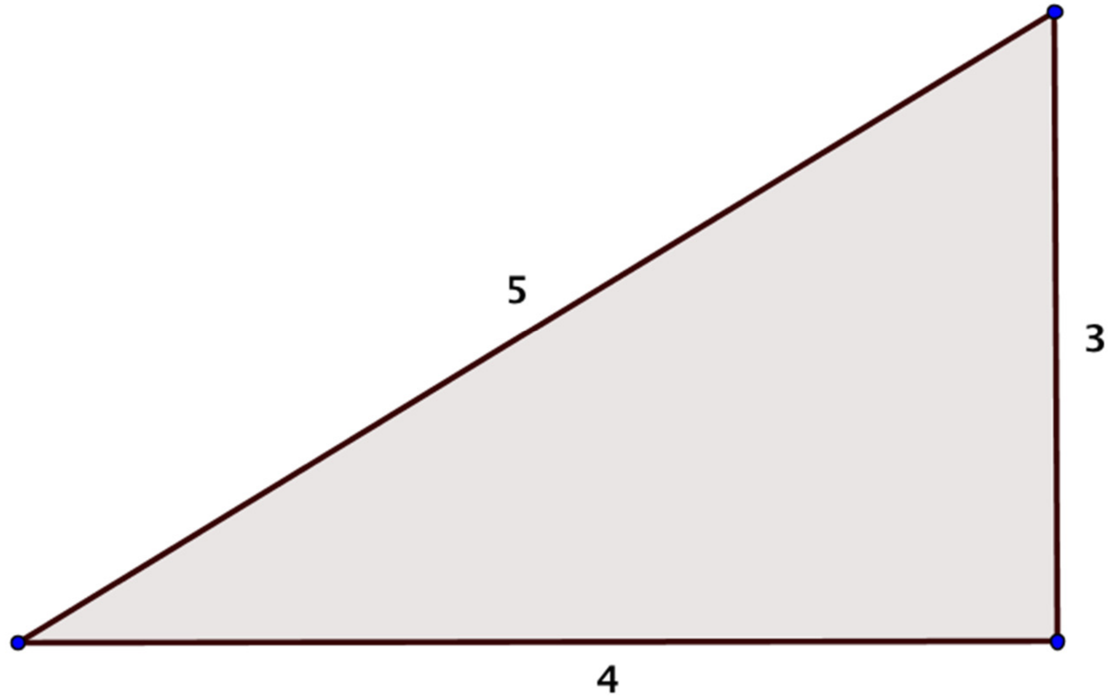
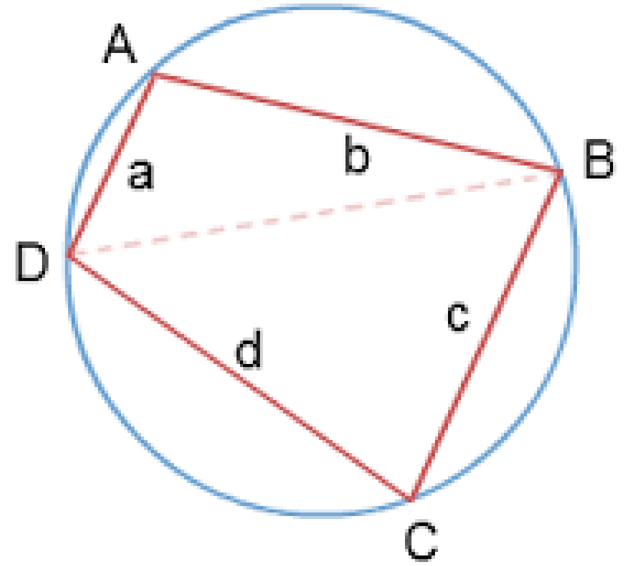
Definition: A *Heron triangle* is one with integer area and side lengths. For instance, this famous triangle:



Question: Is a Heron triangle uniquely determined by the values of its area and perimeter?

Brahmagupta's formula* for the area A of a cyclic quadrilateral with sides a, b, c and d.

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$



where s denotes the semi-perimeter $s = \dfrac{a+b+c+d}{2}$

Now set d = 0 for **Heron's formula** for the area of a triangle

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

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*c.600 AD

We want to study triangles with a given area A and perimeter 2s. But how best to parameterize them?

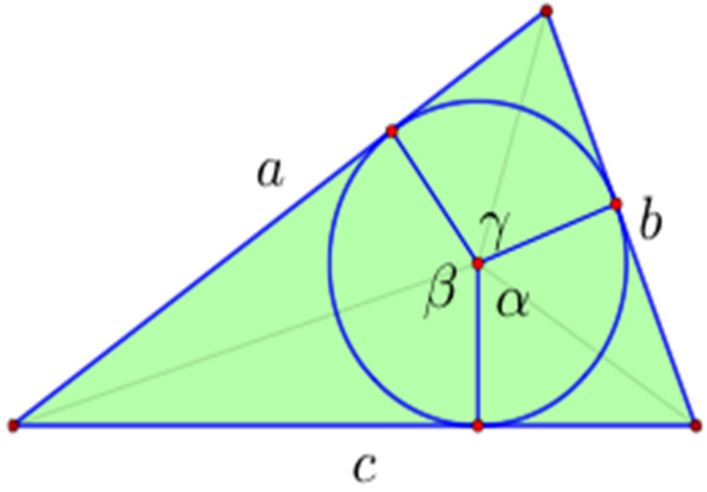
One way would be to think of them as triples (a,b,c) formed from the lengths their sides and regarded as elements in Euclidean 3-space $\mathbf{E}^3$.

The image in $\mathbf{E}^3$ of the set of Heron triangles satisfies a number of constraints (e.g. $a + b > c > 0$. etc) does not appear to have a tractable structure that might give new insights. We therefore look elsewhere.

From the diagram we have

$$A = rs,$$

where r is the radius of the incircle to the triangle.



So all the triangles with given values of A and s have the same incircle and are therefore determined by the angles α, β and γ.

It proves fruitful to parametrize the triangles with a given area A and perimeter $2s$ in terms of these angles.

By elementary trigonometry we have

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\frac{s}{r}=\frac{s^2}{A}=k \text{ say}$$

Now set

$$x=\tan\frac{\alpha}{2},\ y=\tan\frac{\beta}{2} \text{ and } z=\tan\frac{\gamma}{2},$$

Since $\alpha+\beta+\gamma=2\pi$, we have

$$\tan\frac{\gamma}{2}=\tan\left(\pi-\frac{\alpha}{2}-\frac{\beta}{2}\right)=-\frac{x+y}{1-xy}$$

so

$$x+y-\frac{x+y}{1-xy}=k$$

We conclude that

$$x^2y + xy^2 - kxy + k = 0$$

which is an elliptic curve of degree 3 amenable to the advanced methods of algebraic geometry.

Every triangle with a given value of k $(= s^2/A)$ corresponds to a point on this elliptic curve in the region defined by

$$x > 0,\ y > 0 \text{ and } xy > 1$$

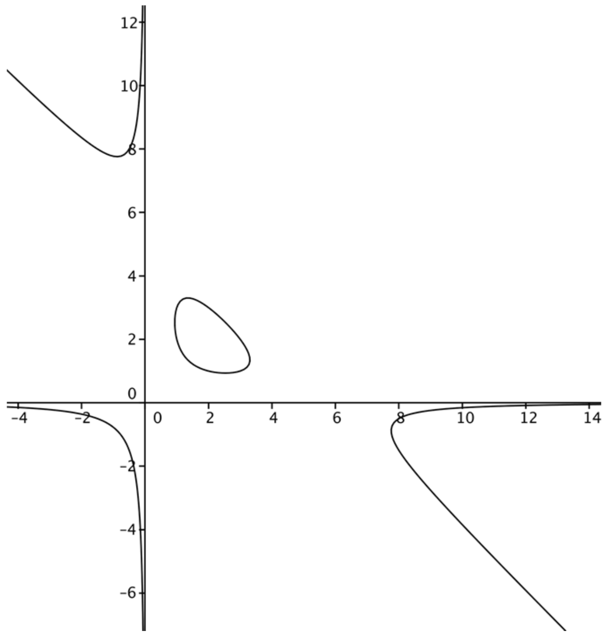
For the (3-4-5) triangle shown at the outset, the radius r of the incircle is 1, k = 6, and the coordinates

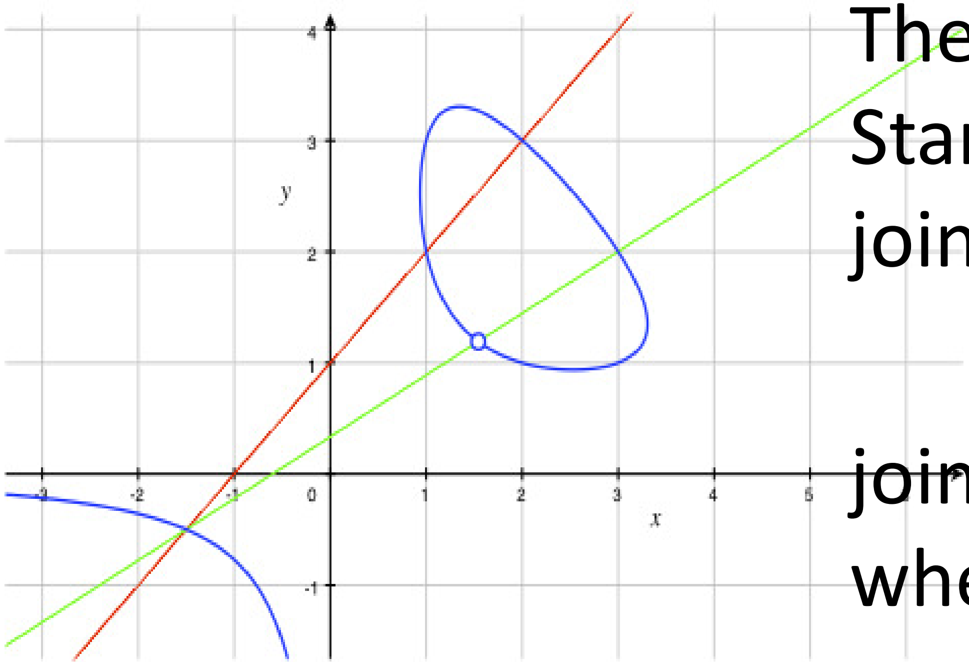
$$x = \tan\frac{\alpha}{2},\ y = \tan\frac{\beta}{2} \text{ and } z = \tan\frac{\gamma}{2}$$

take on the values 1, 2 and 3 in some order.

The elliptic curve ➔

$$x^2y + xy^2 - 6xy + 6 = 0$$





The secant method:
Starting with red line joining the 2 points

(1,2) and (2,3)

join (3,2) to the point where red line meets the curve for a new point⦿ .

The new point $\odot$ we obtain on the elliptic curve has coordinates (54/35, 25/21) and this corresponds to a triangle with sides

41/15, 101/21 and 156/35

which has perimeter 12 and area (by Heron) equal to

$$\sqrt{6(6-(41/15))(6-(101/21))(6-(156/35))} = 6$$

CARRY ON SECANTING

You can create as many new triangles as you like in this way ! The coordinates of each new point involve solving a cubic equation with two known roots.

- Elliptic curves are central to research in number theory. They have applications to the cryptographic schemes behind secure web transactions, and they played a key role in the proof of Fermat's Last Theorem.

- Ronald van Luijk has used deep results in algebraic geometry to prove that there exist infinitely many families, each containing infinitely many triangles with rational sides and rational area and *all with the same perimeter and area.*

12 Triangles found by van Luijk

Perimeter 2s = a + b + c = 6,111,518,179,503,708,972,000 **Area A** = 1,340,792,724,147,847,711,994,993,266,314,426,038,400,000		
a	b	c
1154397878350700583600	2324466316136026062000	2632653985016982326400
1353301222256224441200	2044007602377661720800	2714209354869822810000
1664717974861560418800	1703885276761144351875	2742914927881004201325
1958819929328111850000	1426020908550865426800	2726677341624731695200
2005582596002614412784	1385590865209533198216	2720344718291561361000
2198208931289532607600	1234160196742812482000	2679149051471363882400
2256059203526140412400	1195069414854334519500	2660389561123234040100
2227944754401017652000	1213597769548172408400	2669975655554518911600
2440795514101169425200	1105486738297174396800	2565235927105365150000
2462169105650632177800	1100472310428896790000	2548876763424180004200
2469616851505228370400	1099107024377149242000	2542794303621331359600
2623055767363274578335	1143817472264343917040	2344644939876090476625