Max-plus automata and Tropical identities

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University of Warwick

Birmingham, 15-11-2017

Matrices vs machines...

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Matrices over
$(\mathbb{N} \cup \{-\infty\}, \max, +)$

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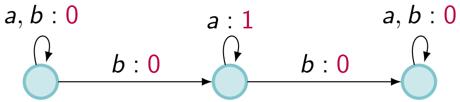
Max-plus Automata



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Set of words A^*: finite sequences of a and b

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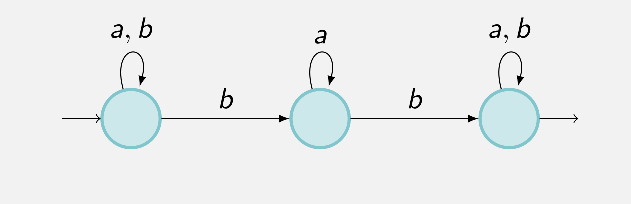

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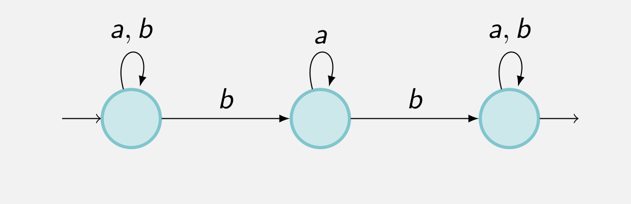

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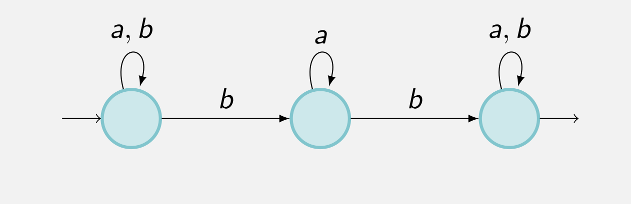

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$\longrightarrow$ Quantitative extension: Weighted automata [Schützenberger, 61]

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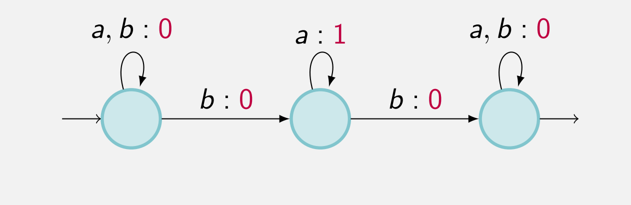

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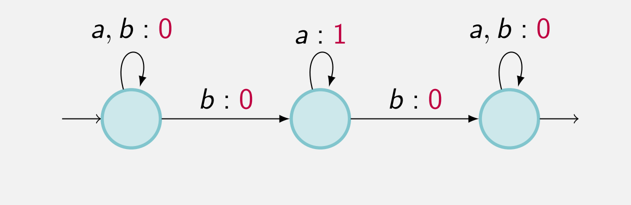

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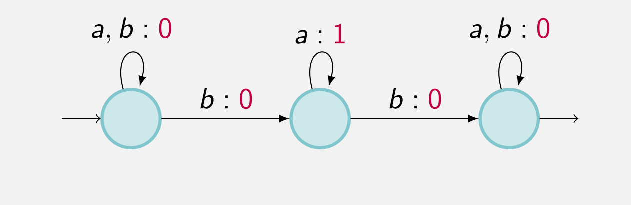

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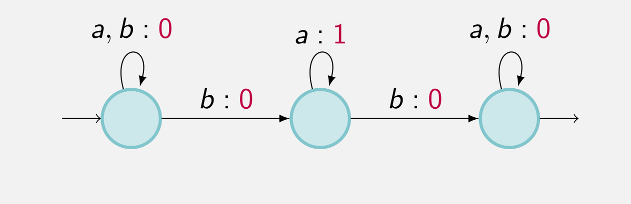

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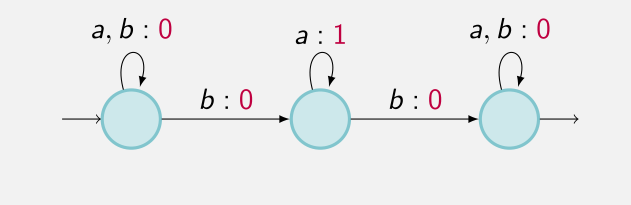
$$a^{n_0} b a^{n_1} b \cdots b a^{n_{k+1}} \mapsto \max(n_1, \ldots, n_k)$$

Matrix representation



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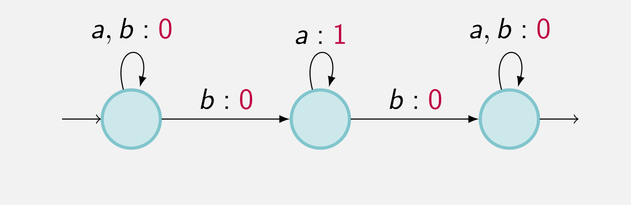

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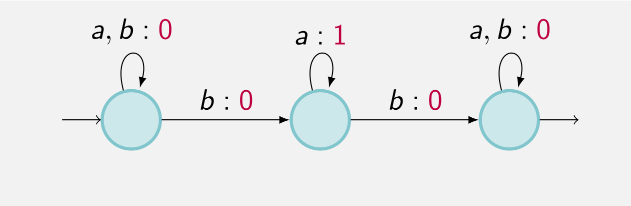

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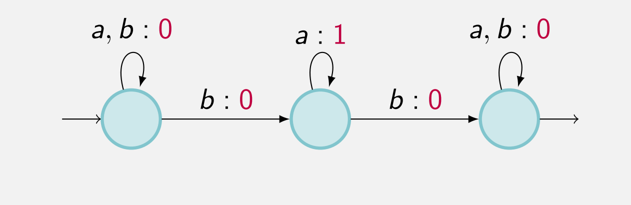

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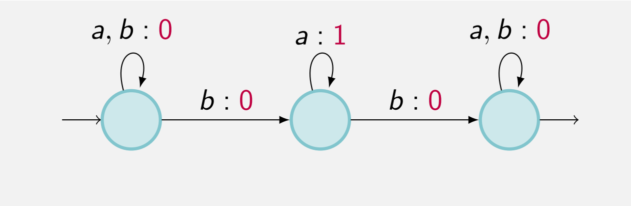

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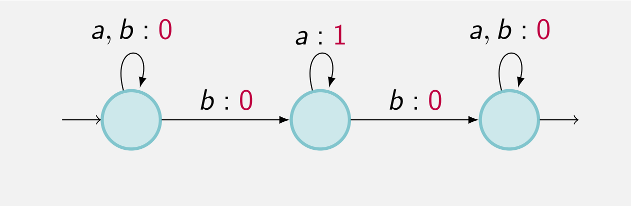

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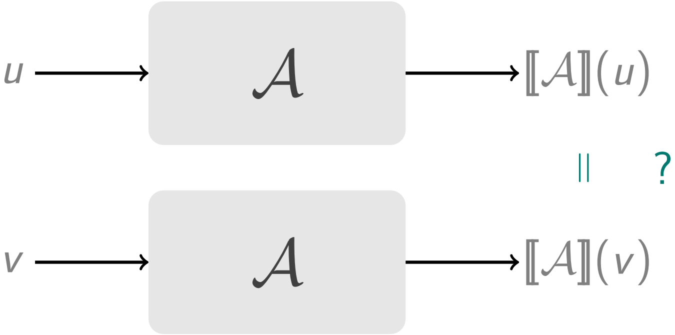
Dimension = Number of states

Questions ?

Decidability and complexity

- Equivalence [Krob]
- Boundedness [Simon]
- Determinisation [Kirsten, Klimann, Lombardy, Mairesse, Prieur]
- Minimisation
- ...

A natural and fundamental question:



Which pairs of inputs can be distinguished
by a given computational model?

Distinguishing words

Semiring $(\mathbb{N} \cup \{-\infty\}, \max, +)$

$[\![\mathcal{A}]\!] : A^* \to \mathbb{N} \cup \{-\infty\}$

$$[\![\mathcal{A}]\!] : w \mapsto \max_{\substack{\rho \text{ accepting path} \\ \text{labelled by } w}} \left(\rho_1 + \rho_2 + \cdots + \rho_{|w|}\right)$$

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$\to$ No

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1. For all $u \neq v$, is there $\mathcal{A} \in \mathcal{C}$ which distinguishes u and v?
   $\to$ Yes
2. Is there $\mathcal{A} \in \mathcal{C}$ which distinguishes all pairs $u \neq v$?
   $\to$ No
3. Minimal size to distinguish two given input words?
   $\to$??????

Given a positive integer n,
are there $u \neq v$ such that
for all max-plus automata $\mathcal{A}$ with at most n states:

$$[\![\mathcal{A}]\!](u) = [\![\mathcal{A}]\!](v) \quad ?$$

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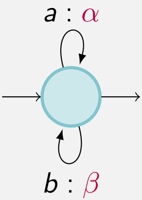
For matrices:
Given a dimension n, does there exists a non trivial identity for the semigroup of square matrices of dimension n ?

If $n = 1$

$A = \{a, b\}$



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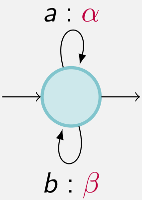

$w \mapsto \alpha|w|_a + \beta|w|_b$

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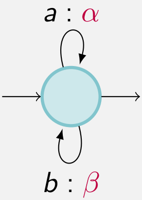

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Max-plus automata with one state can distinguish words with different contents (in particular different lengths), and only these ones.

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2 states [Izhakian, Margolis] - words of length 20

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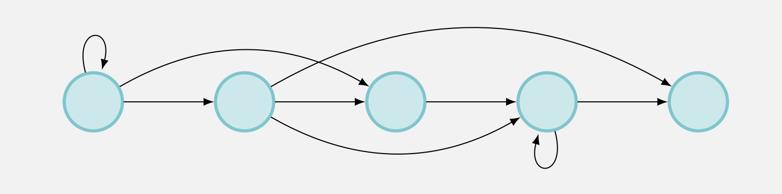
2 states [Izhakian, Margolis] - words of length 20

3 states [Shitov] - words of length 1795308

Triangular automata



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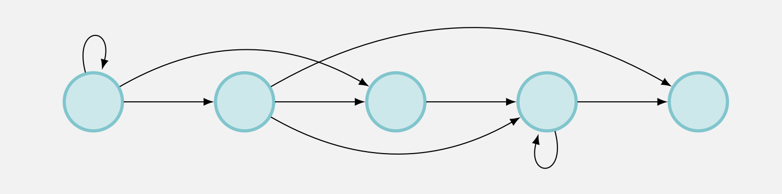

Theorem [Izhakian]

For all n, there exist a pair of distinct words $u \neq v$ such that for all triangular automata $\mathcal{A}$ with at most n states,

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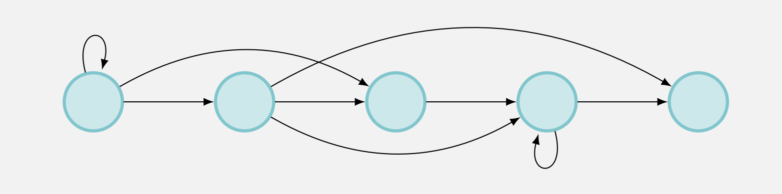

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For $n = 2$, exactly the identities for the bicyclic monoid [D., Johnson, Kambites]

Let's go back to automata with 2 states

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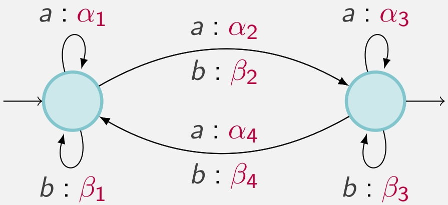

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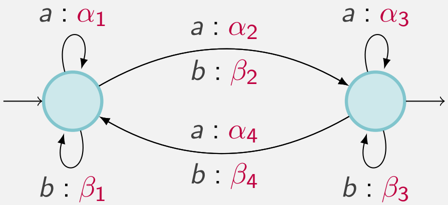

Theorem [D., Johnson] - counter-example to a conjecture of Izhakian

There are two pairs of distinct words of minimal length which cannot be distinguished by any max-plus automata with two states:

$$a^2b^3a^3babab^3a^2 = a^2b^3ababa^3b^3a^2 \text{ and } ab^3a^4baba^2b^3a = ab^3a^2baba^4b^3a$$

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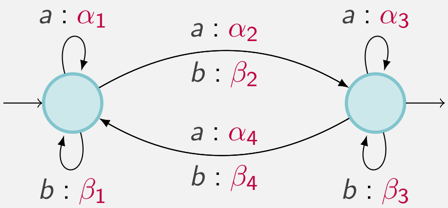

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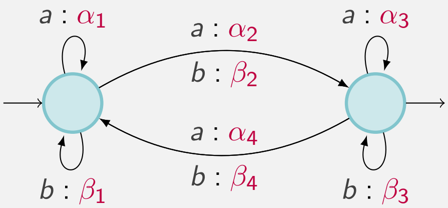

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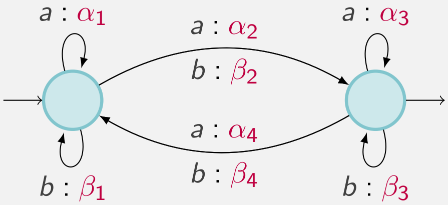

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- $\mathbb{R} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{N}$

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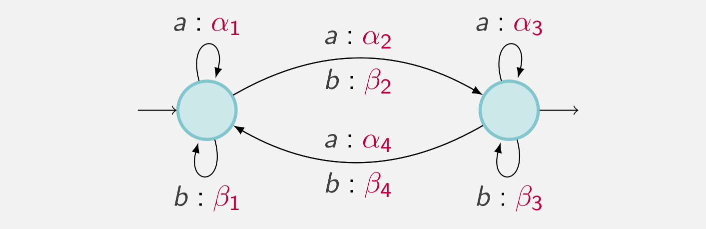

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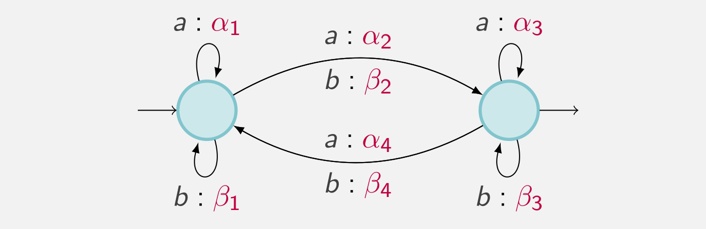

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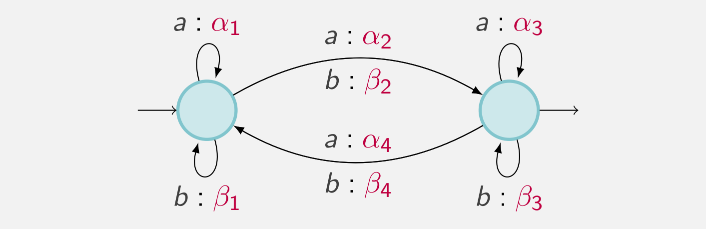

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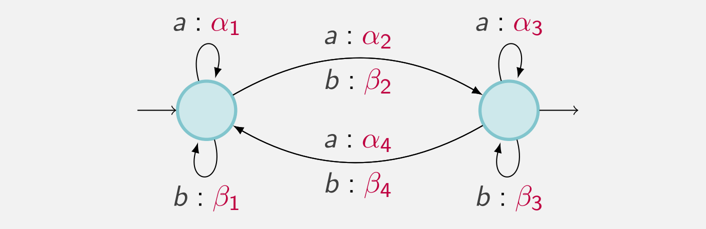

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Second attempt: Give a list of criteria which can be checked

List of criteria

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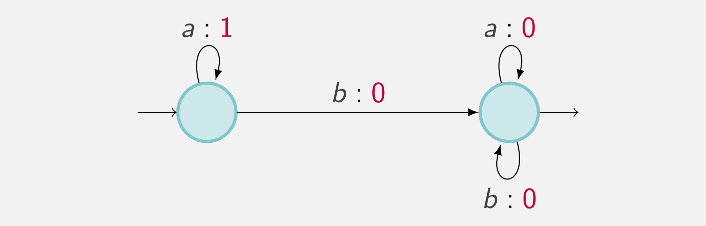
- First and last blocks



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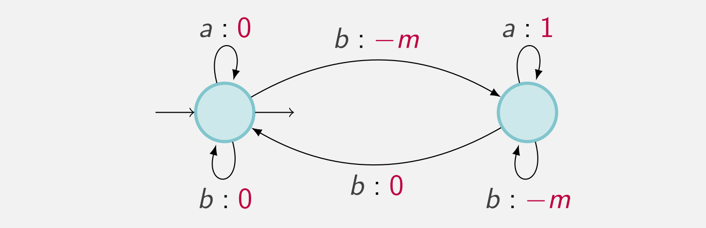

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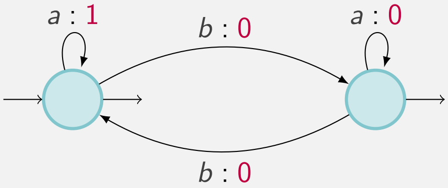
- "Counting modulo 2" criteria

Number of *a*'s after an even number of *b*'s



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- First and last blocks
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- Triangular automata with two states