Tropical Optimization Framework for Analytical Hierarchy Process

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Outline

Introduction: Tropical Optimization

- Tropical (idempotent) mathematics focuses on the theory and applications of semirings with idempotent addition
- The tropical optimization problems are formulated and solved within the framework of tropical mathematics
- Many problems have objective functions defined on vectors over idempotent semifields (semirings with multiplicative inverses)
- The problems find applications in many areas to provide new efficient solutions to various old and novel problems in
  - project scheduling,
  - location analysis,
  - transportation networks,
  - decision making,
  - discrete event systems

Idempotent Algebra: Definitions and Notation

Idempotent Semifield

- *Idempotent semifield:* the algebraic system $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$
- The binary operations $\oplus$ and $\otimes$ are *associative and commutative*
- The carrier set $\mathbb{X}$ has neutral elements, *zero* $\mathbb{0}$ and *identity* $\mathbb{1}$
- Multiplication $\otimes$ is *distributive* over addition
- Addition $\oplus$ is *idempotent:* $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication $\otimes$ is *invertible:* for each nonzero $x \in \mathbb{X}$, there exists an inverse $x^{-1} \in \mathbb{X}$ such that $x \otimes x^{-1} = \mathbb{1}$
- *Algebraic completeness:* the equation $x^p = a$ is solvable for any $a \in \mathbb{X}$ and integer p (there exist powers with rational exponents)
- *Notational convention:* the multiplication sign $\otimes$ will be omitted

Semifield $\mathbb{R}_{\max,\times}$ (Max-Algebra)

- *Definition:* $\mathbb{R}_{\max,\times} = \langle \mathbb{R}_+, 0, 1, \max, \times \rangle$ with $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$
- *Carrier set:* $\mathbb{X} = \mathbb{R}_+$; *zero and identity:* $\mathbb{0} = 0$, $\mathbb{1} = 1$
- *Binary operations:* $\oplus = \max$ and $\otimes = \times$
- *Idempotent addition:* $x \oplus x = \max(x, x) = x$ for all $x \in \mathbb{R}_+$
- *Multiplicative inverse:* for each $x \in \mathbb{R}_+ \setminus \{0\}$, there exists x^{-1}
- *Power notation:* x^y is routinely defined for each $x, y \in \mathbb{R}_+$
- Further examples of real idempotent semifields:

$$\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle,$$
$$\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle,$$
$$\mathbb{R}_{\min,\times} = \langle \mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times \rangle$$

Vector and Matrix Algebra Over $\mathbb{R}_{\max,\times}$

- The scalar idempotent semifield $\mathbb{R}_{\max,\times}$ is routinely extended to idempotent systems of vectors in $\mathbb{R}_{+}^{n}$ and of matrices in $\mathbb{R}_{+}^{m\times n}$
- The matrix and vector operations follow the standard entry-wise formulas with the addition $\oplus = \max$ and the multiplication $\otimes = \times$
- For any vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ in $\mathbb{R}_{+}^{n}$, and a scalar $x \in \mathbb{R}_{+}$, the vector operations follow the conventional rules

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_i = a_i \oplus b_i, \qquad \{x\boldsymbol{a}\}_i = x a_i$$

- For any matrices $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}_{+}^{m\times n}$ and $\boldsymbol{C} = (c_{ij}) \in \mathbb{R}_{+}^{n\times l}$, and $x \in \mathbb{R}_{+}$, the matrix operations are given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} c_{kj}, \quad \{x\boldsymbol{A}\}_{ij} = x a_{ij}$$

Vector and Matrix Algebra Over $\mathbb{R}_{\max,\times}$

- All vectors are column vectors, unless otherwise specified
- The *zero vector* and *vector of ones:*
  $\mathbf{0} = (0,\ldots,0)^T$, $\mathbf{1} = (1,\ldots,1)^T$
- *Multiplicative conjugate transposition* of a nonzero column vector $\boldsymbol{x} = (x_i)$ is the row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \neq 0$, and $x_i^- = 0$ otherwise
- The *zero matrix* and *identity matrix:*
  $\mathbf{0} = (0)$, $\boldsymbol{I} = \mathrm{diag}(1,\ldots,1)$
- *Multiplicative conjugate transposition* of a nonzero matrix $\boldsymbol{A} = (a_{ij})$ is the matrix $\boldsymbol{A}^- = (a_{ij}^-)$, where $a_{ij}^- = a_{ji}^{-1}$ if $a_{ji} \neq 0$, and $a_{ij}^- = 0$ otherwise
- *Integer powers* of square matrices:

$$\boldsymbol{A}^0 = \boldsymbol{I}, \qquad \boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{p-1}, \qquad p \geq 1$$

Square Matrices

- *Trace:* the trace of a matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}_{+}^{n\times n}$ is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$$

- *Eigenvalue:* a scalar λ such that there is a vector $\boldsymbol{x} \neq \boldsymbol{0}$ to satisfy

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

- *Spectral radius:* the maximum eigenvalue given by

$$\rho = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}^{1/m}(\boldsymbol{A}^m)$$

- *Asterate:* the asterate operator (the Kleene star) is given by

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}, \qquad \rho \leq 1$$

Tropical Optimization Problems: Solution Examples

Problem with Pseudo-Quadratic Objective

Given a matrix $\boldsymbol{A} \in \mathbb{R}_{+}^{n\times n}$, find positive vectors $\boldsymbol{x} \in \mathbb{R}_{+}^{n}$ that solve the problem

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

Theorem

Let $\boldsymbol{A}$ be a matrix with tropical spectral radius $\lambda > 0$, and denote $\boldsymbol{B} = (\lambda^{-1}\boldsymbol{A})^{*}$. Then,

- *the minimum of $\boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$ is equal to λ;*
- *all positive solutions are given by*

$$\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0}$$

Maximization Problem with Hilbert (Range, Span) Seminorm

Given a matrix $\boldsymbol{B} \in \mathbb{R}_{+}^{n\times m}$, find positive vectors $\boldsymbol{u} \in \mathbb{R}_{+}^{l}$ that solve the problems

$$\max_{\boldsymbol{u}>\boldsymbol{0}} \quad \boldsymbol{1}^{T}\boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1} \qquad\qquad \min_{\boldsymbol{u}>\boldsymbol{0}} \quad \boldsymbol{1}^{T}\boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$$

Lemma

Let $\boldsymbol{B}$ be a positive matrix, and $\boldsymbol{B}_{lk}$ be the matrix derived from $\boldsymbol{B} = (\boldsymbol{b}_{k})_{k=1}^{m}$ by fixing the entry b_{lk} and replacing the others by 0.

- *The* **maximum** *of $\boldsymbol{1}^{T}\boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$ is equal to $\Delta = \boldsymbol{1}^{T}\boldsymbol{B}\boldsymbol{B}^{-}\boldsymbol{1}$*
- *All positive solutions are given by*

$$\boldsymbol{u} = (\boldsymbol{I} \oplus \boldsymbol{B}_{lk}^{-}\boldsymbol{B})\boldsymbol{v}, \quad \boldsymbol{v} > \boldsymbol{0},$$

where the indices k and l satisfy the condition $\boldsymbol{1}^{T}\boldsymbol{b}_{k}b_{lk}^{-1} = \Delta$

Minimization Problem with Hilbert (Range, Span) Seminorm

Lemma

Let B *be a matrix without zero rows and columns.*

- *The* **minimum** *of* $\mathbf{1}^T \boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\mathbf{1}$ *is equal to* $\Delta = (\boldsymbol{B}(\mathbf{1}^T\boldsymbol{B})^{-})^{-}\mathbf{1}$ *.*
- *Denote by* $\widehat{\boldsymbol{B}}$ *be the sparsified matrix with entries:*

$$\widehat{b}_{ij} = \begin{cases} 0, & \textit{if } b_{ij} < \Delta^{-1}\mathbf{1}^T\boldsymbol{b}_j; \\ b_{ij}, & \textit{otherwise.} \end{cases}$$

Let $\mathcal{B}$ *be the set of matrices obtained from* $\widehat{\boldsymbol{B}}$ *by fixing one nonzero entry in each row and setting the others to* 0 *.*
Then, all positive solutions are given by

$$\boldsymbol{u} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{B}_1^{-}\mathbf{1}\mathbf{1}^T\boldsymbol{B})\boldsymbol{v}, \quad \boldsymbol{v} > \mathbf{0}, \quad \boldsymbol{B}_1 \in \mathcal{B}$$

Analytical Hierarchy Process: Traditional Approach

Pairwise Comparison

- Given m criteria and n choices, the problem is to find priorities of choices from pairwise comparisons of criteria and of choices
- Outcome of comparison is given by a matrix $\boldsymbol{A} = (a_{ij})$, where a_{ij} shows the relative priority of choice i over j
- Note that $a_{ij} = 1/a_{ji} > 0$
- Scale (Saaty, 2005):

a_{ij}	Meaning
1	i equally important as j
3	i moderately more important than j
5	i strongly more important than j
7	i very strongly more important than j
9	i extremely more important than j

Consistency

- A pairwise comparison matrix $\boldsymbol{A}$ is consistent if its entries are transitive to satisfy the condition $a_{ij} = a_{ik}a_{kj}$ for all i, j, k
- Each consistent matrix $\boldsymbol{A}$ has unit rank and is given by $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{x}^{T}$, where $\boldsymbol{x}$ is a positive vector that entirely specifies $\boldsymbol{A}$
- If a comparison matrix $\boldsymbol{A}$ is consistent, the vector $\boldsymbol{x}$ represents, up to a positive factor, the individual priorities of choices
- Since the comparison matrices are usually inconsistent, a problem arises to approximate these matrices by consistent matrices

Principal Eigenvector Method and Weighted Sum Solution

- The traditional AHP uses approximation of pairwise comparison matrices by consistent matrices with the principal eigenvectors
- Let $\boldsymbol{A}_0$ be a matrix of pairwise comparison of criteria, and $\boldsymbol{A}_k$ be a matrix of pairwise comparison of choices for criterion k
- Let $\boldsymbol{w} = (w_k)_{k=1}^m$ be the principal eigenvector of $\boldsymbol{A}_0$: the vector of priorities (weights) for criteria
- Let $\boldsymbol{x}_k$ be the principal eigenvector of $\boldsymbol{A}_k$: the vector of priorities of choices with respect to criterion k
- The resulting vector $\boldsymbol{x}$ of priorities of choices is calculated as

$$\boldsymbol{x} = \sum_{k=1}^{m} w_k \boldsymbol{x}_k$$

Minimax approximation based AHP

Log-Chebyshev Approximation of Comparison Matrices

- Consider the problem to approximate a pairwise comparison matrix $\boldsymbol{A} = (a_{ij})$ by a consistent matrix $\boldsymbol{X} = (x_{ij})$, where

$$a_{ij} = 1/a_{ji}, \qquad x_{ij} = x_i/x_j$$

- The log-Chebyshev distance between $\boldsymbol{A}$ and $\boldsymbol{X}$ is defined as

$$\max_{1 \leq i,j \leq n} |\log a_{ij} - \log x_{ij}| = \log \max_{1 \leq i,j \leq n} \max \left(\frac{a_{ij}}{x_{ij}}, \frac{x_{ij}}{a_{ij}} \right)$$

- Minimizing the log-Chebyshev distance is equivalent to minimizing

$$\max_{1 \leq i,j \leq n} \max \left(\frac{a_{ij}}{x_{ij}}, \frac{x_{ij}}{a_{ij}} \right) = \max_{1 \leq i,j \leq n} \max \left(\frac{a_{ij} x_j}{x_i}, \frac{a_{ji} x_i}{x_j} \right) = \max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i}$$

Approximation as Tropical Optimization Problem

- Tropical representation of the objective function in terms of $\mathbb{R}_{\max,\times}$

$$\max_{1\le i,j\le n} \frac{a_{ij}x_j}{x_i} = \bigoplus_{1\le i,j\le n} x_i^{-1}a_{ij}x_j = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

- In the framework of the idempotent semifield $\mathbb{R}_{\max,\times}$, the minimax approximation problem takes the form

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

Theorem

Let $\boldsymbol{A}$ be a pairwise comparison matrix with tropical spectral radius λ , and $\boldsymbol{B} = (\lambda^{-1}\boldsymbol{A})^{*}$. Then,

- *all priority vectors are given by*

$$\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0}$$

Weighted Approximation Under Several Criteria

- Simultaneous minimax approximation of the matrices $\boldsymbol{A}_k = (a_{ij}^{(k)})$ with weights $w_k > 0$ by a consistent matrix involves minimizing

$$\max_{1\leq k\leq m} w_k \left(\max_{1\leq i,j\leq n} (a_{ij}^{(k)} x_j)/x_i \right) = \max_{1\leq i,j\leq n} \max_{1\leq k\leq m} (w_k a_{ij}^{(k)}) x_j/x_i.$$

- In terms of $\mathbb{R}_{\max,\times}$, the approximation problem takes the form

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \quad \boldsymbol{x}^{-}(w_1\boldsymbol{A}_1 \oplus \cdots \oplus w_m\boldsymbol{A}_m)\boldsymbol{x}$$

Theorem

Let $\boldsymbol{A}_1,\ldots,\boldsymbol{A}_m$ be comparison matrices, $w_1,\ldots,w_m$ be weights, $\boldsymbol{C} = w_1\boldsymbol{A}_1 \oplus \cdots \oplus w_m\boldsymbol{A}_m$ be a matrix with tropical spectral radius μ, and $\boldsymbol{B} = (\mu^{-1}\boldsymbol{C})^*$. Then,

- *all priority vectors are given by*

$$\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0}$$

Most and Least Differentiating Priority Vectors

- The priority vectors $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}$ obtained by the minimax approximation may be not unique up to a positive factor
- Further analysis is then needed to reduce to a few representative solutions, such as some “best” and “worst” priority vectors
- One can take two vectors that most and least differentiate between the choices with the highest and lowest priorities
- The most and least differentiating priority vectors are obtained by maximizing and minimizing the contrast ratio

$$\max_{1\leq i\leq n} x_i / \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} x_i \times \max_{1\leq i\leq n} x_i^{-1}$$

- In terms of the semifield $\mathbb{R}_{\max,\times}$, the contrast ratio is written as

$$\boldsymbol{1}^T \boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{1} = \boldsymbol{1}^T \boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$$

Most Differentiating Priority Vector

- In terms of the semifield $\mathbb{R}_{\max,\times}$, the problem to maximize the contrast ratio is written as

$$\max_{\boldsymbol{u}>\boldsymbol{0}} \quad \boldsymbol{1}^T \boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$$

- If u is a solution, then $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}$ is the most differentiating vector

Most Differentiating Priority Vector

Theorem

Let $\boldsymbol{B}$ *be a matrix defining a set of priority vectors* $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}$ *,* $\boldsymbol{u} > \boldsymbol{0}$ *, and* $\boldsymbol{B}_{lk}$ *be the matrix obtained from* $\boldsymbol{B} = (\boldsymbol{b}_j)$ *by fixing the entry* b_{lk} *and replacing the others by* 0 *.*

- *The* **maximum** *of* $\boldsymbol{1}^T\boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$ *is equal to* $\Delta = \boldsymbol{1}^T\boldsymbol{B}\boldsymbol{B}^{-}\boldsymbol{1}$
- *The* **most differentiating priority vectors** *are given by*

$$\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{I} \oplus \boldsymbol{B}_{lk}^{-}\boldsymbol{B})\boldsymbol{v}, \quad \boldsymbol{v} > \boldsymbol{0},$$

where the indices k *and* l *satisfy the condition* $\boldsymbol{1}^T\boldsymbol{b}_k b_{lk}^{-1} = \Delta$

Least Differentiating Priority Vector

- In terms of the semifield $\mathbb{R}_{\max,\times}$, the problem to minimize the contrast ratio is written as

$$\min_{\boldsymbol{u}>\boldsymbol{0}} \quad \boldsymbol{1}^T\boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\boldsymbol{1}$$

- If $\boldsymbol{u}$ is a solution, then $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}$ is the least differentiating vector

Least Differentiating Priority Vector

Theorem

Let B be a matrix defining a set of priority vectors $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}$, $\boldsymbol{u} > \boldsymbol{0}$.

- *The* **minimum** *of $\mathbf{1}^T \boldsymbol{B}\boldsymbol{u}(\boldsymbol{B}\boldsymbol{u})^{-}\mathbf{1}$ is equal to $\Delta = (\boldsymbol{B}(\mathbf{1}^T \boldsymbol{B})^{-})^{-}\mathbf{1}$.*
- *Denote by $\widehat{\boldsymbol{B}}$ be the sparsified matrix with entries:*

$$\widehat{b}_{ij} = \begin{cases} 0, & \text{if } b_{ij} < \Delta^{-1}\mathbf{1}^T \boldsymbol{b}_j; \\ b_{ij}, & \text{otherwise.} \end{cases}$$

Let $\mathcal{B}$ be the set of matrices obtained from $\widehat{\boldsymbol{B}}$ by fixing one nonzero entry in each row and setting the others to 0.
Then the **least differentiating priority vectors** *are given by*

$$\boldsymbol{u} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{B}_1^{-}\mathbf{1}\mathbf{1}^T \boldsymbol{B})\boldsymbol{v}, \quad \boldsymbol{v} > \boldsymbol{0}, \quad \boldsymbol{B}_1 \in \mathcal{B}$$

Illustrative Example: Selecting Plan for Vacation

Problem: Select a Place to Spend a Week (Saaty, 1977)

- Five criteria: (1) cost of the trip from Philadelphia, (2) sight-seeing opportunities, (3) entertainment (doing things), (4) way of travel, (5) eating places; with the criteria comparison matrix

$$\boldsymbol{A}_0 = \begin{pmatrix} 1 & 1/5 & 1/5 & 1 & 1/3 \\ 5 & 1 & 1/5 & 1/5 & 1 \\ 5 & 5 & 1 & 1/5 & 1 \\ 1 & 5 & 5 & 1 & 5 \\ 3 & 1 & 1 & 1/5 & 1 \end{pmatrix}$$

- Four places: (1) short trips from Philadelphia (i.e., New York, Washington, Atlantic City, New Hope, etc.), (2) Quebec, (3) Denver, (4) California

Problem (cont.)

- Pairwise comparison matrices of places with respect to criteria

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 3 & 7 & 9 \\ 1/3 & 1 & 6 & 7 \\ 1/7 & 1/6 & 1 & 3 \\ 1/9 & 1/7 & 1/3 & 1 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 1 & 1/5 & 1/6 & 1/4 \\ 5 & 1 & 2 & 4 \\ 6 & 1/2 & 1 & 6 \\ 4 & 1/4 & 1/6 & 1 \end{pmatrix},$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 1 & 7 & 7 & 1/2 \\ 1/7 & 1 & 1 & 1/7 \\ 1/7 & 1 & 1 & 1/7 \\ 2 & 7 & 7 & 1 \end{pmatrix}, \qquad \boldsymbol{A}_4 = \begin{pmatrix} 1 & 4 & 1/4 & 1/3 \\ 1/4 & 1 & 1/2 & 3 \\ 4 & 2 & 1 & 3 \\ 3 & 1/3 & 1/3 & 1 \end{pmatrix},$$

$$\boldsymbol{A}_5 = \begin{pmatrix} 1 & 1 & 7 & 4 \\ 1 & 1 & 6 & 3 \\ 1/7 & 1/6 & 1 & 1/4 \\ 1/4 & 1/3 & 4 & 1 \end{pmatrix}$$

Solution: Evaluating Priority Vector (Weights) for Criteria

- The tropical spectral radius of the comparison matrix $\boldsymbol{A}_0$

$$\lambda = (a_{14}a_{43}a_{32}a_{21})^{1/4} = 5^{3/4}$$

- The Kleene star of the matrix $\lambda^{-1}\boldsymbol{A}_0$

$$(\lambda^{-1}\boldsymbol{A}_0)^* = \boldsymbol{I} \oplus \lambda^{-1}\boldsymbol{A}_0 \oplus \lambda^{-2}\boldsymbol{A}_0^2 \oplus \lambda^{-3}\boldsymbol{A}_0^3 \oplus \lambda^{-4}\boldsymbol{A}_0^4$$

$$= \begin{pmatrix} 1 & 5^{-1/4} & 5^{-1/2} & 5^{-3/4} & 5^{-1/2} \\ 5^{1/4} & 1 & 5^{-1/4} & 5^{-1/2} & 5^{-1/4} \\ 5^{1/2} & 5^{1/4} & 1 & 5^{-1/4} & 1 \\ 5^{3/4} & 5^{1/2} & 5^{1/4} & 1 & 5^{1/4} \\ 3 \cdot 5^{-3/4} & 3 \cdot 5^{-1} & 3 \cdot 5^{-5/4} & 3 \cdot 5^{-3/2} & 3 \cdot 5^{-5/4} \end{pmatrix}$$

- The priority (weight) vector for criteria (pseudo-quadratic problem)

$$\boldsymbol{w} = (1,\ 5^{1/4},\ 5^{1/2},\ 5^{3/4},\ 3 \cdot 5^{-3/4})$$

Derivation of All Priority Vectors for Places

- The weighted combination of comparison matrices of places

$$\boldsymbol{C} = \boldsymbol{A}_1 \oplus 5^{1/4}\boldsymbol{A}_2 \oplus 5^{1/2}\boldsymbol{A}_3 \oplus 5^{3/4}\boldsymbol{A}_4 \oplus (3 \cdot 5^{-3/4})\boldsymbol{A}_5$$
$$= \begin{pmatrix} 5^{3/4} & 7 \cdot 5^{1/2} & 7 \cdot 5^{1/2} & 9 \\ 5^{5/4} & 5^{3/4} & 6 & 3 \cdot 5^{3/4} \\ 4 \cdot 5^{3/4} & 2 \cdot 5^{3/4} & 5^{3/4} & 3 \cdot 5^{3/4} \\ 3 \cdot 5^{3/4} & 7 \cdot 5^{1/2} & 7 \cdot 5^{1/2} & 5^{3/4} \end{pmatrix}$$

- The tropical spectral radius of matrix $\boldsymbol{C}$

$$\mu = (c_{13}c_{31})^{1/2} = 2 \cdot 5^{5/8}7^{1/2}$$

Derivation of All Priority Vectors for Places (cont.)

- The Kleene star of matrix $\mu^{-1}\boldsymbol{C}$

$$(\mu^{-1}\boldsymbol{C})^{*} = \boldsymbol{I} \oplus \mu^{-1}\boldsymbol{C} \oplus \mu^{-2}\boldsymbol{C}^{2} \oplus \mu^{-3}\boldsymbol{C}^{3}$$

$$= \begin{pmatrix} 1 & r/4 & r/4 & 3/4 \\ 3/r & 1 & 3/4 & 3/r \\ 4/r & 1 & 1 & 3/r \\ 1 & r/4 & r/4 & 1 \end{pmatrix}, \qquad r = 2 \cdot 7^{1/2} 5^{-1/8} \approx 4.33$$

- All solution vectors (pseudo-quadratic problem)

$$\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{B} = \begin{pmatrix} 1 & r/4 & 3/4 \\ 3/r & 1 & 3/r \\ 4/r & 1 & 3/r \\ 1 & r/4 & 1 \end{pmatrix}, \qquad \boldsymbol{u} > \boldsymbol{0}$$

Evaluation of Most Differentiating Solutions

- The vectors with maximum differentiation between choices of lowest and highest priorities (maximization of Hilbert seminorm)

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 3/r \\ 4/r \\ 1 \end{pmatrix} u, \quad \boldsymbol{x}_2 = \begin{pmatrix} 3/4 \\ 3/r \\ 3/r \\ 1 \end{pmatrix} v, \quad u, v > 0, \quad r = 2{\cdot}7^{1/2}5^{-1/8} \approx 4.33$$

- Examples of vectors with $u = v = 1$

$$\boldsymbol{x}_1 \approx (1.00, 0.69, 0.92, 1.00)^T, \qquad \boldsymbol{x}_2 \approx (0.75, 0.69, 0.69, 1.00)^T$$

- The priority order of places according to the vectors

$$(4) \equiv (1) \succ (3) \succ (2), \qquad (4) \succ (1) \succ (3) \equiv (2)$$

Evaluation of Least Differentiating Solutions

- The vectors with minimum differentiation between choices of lowest and highest priorities (minimization of Hilbert seminorm)

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 4/r \\ 4/r \\ 1 \end{pmatrix} u, \qquad u > 0, \quad r = 2 \cdot 7^{1/2} 5^{-1/8} \approx 4.33$$

- Example of vectors with $u = 1$ and related priority order

$$\boldsymbol{x}_1 \approx (1.00, 0.92, 0.92, 1.00)^T, \qquad (4) \equiv (1) \succ (3) \equiv (2)$$

- Combined new orders versus order by (Saaty, 1977)

NEW: $(4) \succeq (1) \succ (3) \succeq (2)$ OLD: $(1) \succ (3) \succ (4) \succ (2)$

Concluding Remarks

- We have proposed a new implementation of the AHP method, based on minimax approximation and tropical optimization
- The new AHP implementation uses log-Chebyshev matrix approximation instead of the principal eigenvector method
- The weights of criteria are incorporated into the evaluation of the priorities of choices rather then used to form the result directly

Concluding Remarks

- Since the solution obtained is usually non-unique, a technique has been proposed to find two representative priority vectors
- As such solutions, those vectors are taken which most and least differentiate between choices with the highest and lowest priorities
- The above problems have been formulated in the framework of tropical mathematics, and solved as tropical optimization problems
- Exact solutions to the problems have been given in a compact vector form ready for further analysis and practical implementation