Condition numbers in nonarchimedean semidefinite programming . . . and what they say about stochastic mean payoff games

Xavier Allamigeon, Stéphane Gaubert, Ricardo Katz, Mateusz Skomra

INRIA and CMAP, École polytechnique, CNRS

January 24, 2019, Birmingham

Feasibility semidefinite programming problem

Definition (spectrahedron)

Given symmetric matrices $Q^{(0)}, \ldots, Q^{(n)} \in \mathbb{R}^{m \times m}$, the associated spectrahedron is defined as

$$S = \{ x \in \mathbb{R}^n : Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \text{ is positive semidefinite} \}.$$
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The semidefinite feasibility problem (SFDP) consists in deciding whether $S = \emptyset$. 

Allamigeon, Gaubert, Katz, Skomra (Inria-X)  
Nonarchimedean SDP  
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- The **semidefinite feasibility problem** (SFDP) consists in deciding whether $S = \varnothing$.
- The **semidefinite programming** problem (SDP) consists in minimizing a linear form over $S$. 
- SDP can be solved in polynomial time by the ellipsoid or interior point methods in a restricted sense.
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We obtain $\varepsilon$-approximate solutions. Complexity bounds:

$$\text{Poly}(n, m, \log \varepsilon, \log R, \log r, \ldots),$$

where $(R, r, \ldots)$ are metric estimates of the spectrahedron ($\log R$ can be exponential in $n$).
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use some metric geometry ideas
A (formal generalized) Puiseux series is a series of form

$$x = x(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i},$$

where the sequence $\alpha_i \in \mathbb{R}$ is strictly decreasing and either finite or unbounded and $c_i$ are real. Includes (generalized) Dirichlet series $\alpha_i = -\log i$, $t = \exp(s)$. Hardy, Riesz 1915

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- We say that \(x \succeq y\) if \(x(t) \succeq y(t)\) for all \(t\) large enough. This is a linear order on \(K\).

Definition (SDFP over Puiseux series)

Given symmetric matrices $Q^{(0)}, Q^{(1)}, \ldots, Q^{(n)}$, denote

$$Q(x) = Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)}.$$ 

Decide if the following spectrahedron is empty

$$\mathcal{S} = \{ x \in \mathbb{K}^n_{\geq 0} : Q(x) \text{ is positive semidefinite} \}$$
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Proposition

$\mathcal{S} \neq \emptyset$ iff for all $t$ large enough, the following real spectrahedron is non-empty

$$\mathcal{S}(t) = \{x \in \mathbb{R}^n_{\geq 0} : Q^{(0)}(t) + x_1 Q^{(1)}(t) + \cdots + x_n Q^{(n)}(t) \text{ is pos. semidef.}\}$$

Proof. $K$ is the field of germs of univariate functions definable in a o-minimal structure.
Theorem (Allamigeon, SG, Skomra)

There is a correspondence between nonarchimedean semidefinite programming problems and zero-sum stochastic games with perfect information. If the valuations of the matrices \( Q^{(i)} \) are generic, feasibility holds iff Player Max wins the game.
Take the spectrahedral cone

\[ Q(x) := \begin{bmatrix} tx_3 & -x_1 & -t^{3/4}x_3 \\ -x_1 & t^{-1}x_1 + t^{-5/4}x_3 - x_2 & -x_3 \\ -t^{3/4}x_3 & -x_3 & t^{9/4}x_2 \end{bmatrix} \succeq 0. \]
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- Circles: Min plays, Square: Max plays, Bullet: Nature flips coin, Payments made by Min to Max.
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- Max is winning implies that the cone is nontrivial, and yields a feasible point \((t^{1.06}, t^{0.02}, t^{1.13})\).
Benchmark

We tested our method on randomly chosen matrices $Q^{(1)}, \ldots, Q^{(n)} \in \mathbb{K}^{m \times m}$ with positive entries on diagonals and no zero entries. We used the value iteration algorithm.

<table>
<thead>
<tr>
<th>$(n, m)$</th>
<th>(50, 10)</th>
<th>(50, 40)</th>
<th>(50, 50)</th>
<th>(50, 100)</th>
<th>(50, 1000)</th>
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<td>0.000049</td>
<td>0.000077</td>
<td>0.000279</td>
<td>0.026802</td>
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<table>
<thead>
<tr>
<th>$(n, m)$</th>
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<th>(100, 15)</th>
<th>(100, 80)</th>
<th>(100, 100)</th>
<th>(100, 1000)</th>
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<tbody>
<tr>
<td>time</td>
<td>0.000025</td>
<td>0.000270</td>
<td>0.000366</td>
<td>0.000656</td>
<td>0.053944</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>$(n, m)$</th>
<th>(1000, 10)</th>
<th>(1000, 50)</th>
<th>(1000, 100)</th>
<th>(1000, 200)</th>
<th>(1000, 500)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>0.000233</td>
<td>0.073544</td>
<td>0.015305</td>
<td>0.027762</td>
<td>0.148714</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(n, m)$</th>
<th>(2000, 10)</th>
<th>(2000, 70)</th>
<th>(2000, 100)</th>
<th>(10000, 150)</th>
<th>(10000, 400)</th>
</tr>
</thead>
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<tr>
<td>time</td>
<td>0.000487</td>
<td>1.852221</td>
<td>0.087536</td>
<td>19.919844</td>
<td>2.309174</td>
</tr>
</tbody>
</table>

Table: Execution time (in sec.) of Procedure **CheckFeasibility** on random instances.
Experimental phase transition for random nonarchimedean SDP

\[ n = \text{\# variables, } m = \text{size matrices} \]
The present work on tropical condition numbers grew to explain this picture.
Valuation of Puiseux series

\[ x = x(t) = \sum_{k=1}^{\infty} c_k t^{\alpha_k} \]

\[ \text{val}(x) = \lim_{t \to \infty} \frac{\log |x(t)|}{\log t} = \alpha_1 \quad (\text{and val}(0) = -\infty). \]
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Lemma

Suppose that \( x, y \in K_{\geq 0}^n \). Then

- \( x \succeq y \implies \text{val}(x) \geq \text{val}(y) \)
- \( \text{val}(x + y) = \max(\text{val}(x), \text{val}(y)) \)
- \( \text{val}(xy) = \text{val}(x) + \text{val}(y) \).
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- \( \text{val}(xy) = \text{val}(x) + \text{val}(y) \).

Thus, \( \text{val} \) is a morphism from \( \mathbb{K}_n^{\geq 0} \) to a semifield of characteristic one, the tropical semifield \( \mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +) \).
Tropical spectrahedra

Definition

Suppose that $\mathcal{S}$ is a spectrahedron in $K^n_{\geq 0}$. Then we say that $\text{val}(\mathcal{S})$ is a tropical spectrahedron.

How can we study these creatures?
A $S \subset \mathbb{K}^n$ is basic semialgebraic if

$$S = \{(x_1, \ldots, x_n) \in \mathbb{K}^n : P_i(x_1, \ldots, x_n) \circ 0, \circ \in \{>, =\}, \forall i \in [q]\}$$

where $P_1, \ldots, P_q \in \mathbb{K}[x_1, \ldots, x_n]$. A semialgebraic set is a finite union of basic semialgebraic sets.
A \( S \subset \mathbb{K}^n \) is basic semialgebraic if

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A set \( S \subset \mathbb{R}^n \) is basic semilinear if it is of the form

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where \( \ell_1, \ldots, \ell_q \) are linear forms with integer coefficients, \( h^{(1)}, \ldots, h^{(q)} \in \mathbb{R} \). A semilinear set is a finite union of basic semilinear sets.
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**Theorem (Alessandrini, Adv. in Geom. 2013)**

*If $S \subset \mathbb{K}^n_{>0}$ is semi-algebraic, then $\text{val}(S) \subset \mathbb{R}^n$ is semilinear and it is closed.*
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If $S \subset \mathbb{K}^n_{>0}$ is semi-algebraic, then $\text{val}(S) \subset \mathbb{R}^n$ is semilinear and it is closed.

\( S := \text{val}(\mathcal{S}) \) is tropically convex

\[
\max(\alpha, \beta) = 0, \ u, v \in S \implies \sup(\alpha e + u, \beta e + v) \in S
\]

where \( e = (1, \ldots, 1)^T \).
$\mathcal{S} := \text{val}(\mathcal{S})$ is tropically convex

$$\max(\alpha, \beta) = 0, \ u, v \in \mathcal{S} \implies \sup(\alpha e + u, \beta e + v) \in \mathcal{S},$$

where $e = (1, \ldots, 1)^\top$.

**Figure:** Tropical spectrahedron.
Theorem (Semi-algebraic version of Kapranov theorem, Allamigeon, SG, Skomra arXiv:1610.06746)

Consider a collection of \( m \) regions delimited by hypersurfaces:

\[
S_i := \{ x \in \mathbb{K}_\geq 0^n \mid P_i^-(x) \leq P_i^+(x) \}, \quad i \in [m]
\]

where \( P_i^\pm = \sum_{\alpha} p_{i,\alpha}^\pm x^\alpha \in \mathbb{K}_\geq 0[x] \), and let

\[
S_i := \{ x \in \mathbb{R}^n \mid \max_{\alpha}(\text{val } p_{i,\alpha}^- + \langle \alpha, x \rangle) \leq \max_{\alpha}(\text{val } p_{i,\alpha}^+ + \langle \alpha, x \rangle) \}
\]

Then

\[
\text{val}(\bigcap_{i \in [m]} S_i) \subset \bigcap_{i \in [m]} \text{val}(S_i) \subset \bigcap_{i \in [m]} S_i
\]

and the equality holds if \( \bigcap_{i \in [m]} S_i \) is the closure of its interior; in particular if the valuations \( \text{val } p_{i,\alpha}^\pm \) are generic.
Example 1.

\[ S = \{ x \in \mathbb{K}_0^3 \mid x_1^2 \leq tx_2 + t^4x_2x_3 \} \]

\[ \text{val } S = \{ x \in \mathbb{R}^3 \mid 2x_1 \leq \max(1 + x_2, 4 + x_2 + x_3) \} \]

Example 2.

**Figure:** This set is the closure of its interior.
The correspondence between stochastic mean payoff games and nonarchimedean spectrahedra explained
Stochastic mean payoff games

Two player, **Min** and **Max**, and a half-player, **Nature**, move a token on a digraph, alternating moves in a cyclic way:
Stochastic mean payoff games

Two player, Min and Max, and a half-player, Nature, move a token on a digraph, alternating moves in a cyclic way:

- If the current state $i$ belongs to Player Min, this player chooses an arc $i \rightarrow j$, and receives $A_{ji}$ from Player Max.
-
- The current state $j$ now belongs to the half-player Nature, Nature throws a dice and next state becomes $r$ with probability $P_{jr}$.
-
- The current state $r$ now belongs to Player Max, this player chooses an arc $r \rightarrow s$, and receives $B_{rs}$ from Player Max.
-
- The current state $s$ now belongs to Player Min, and so on.
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- The current state $r$ now belongs Player Max, this player chooses an arc $r \rightarrow s$, and receives $B_{rs}$ from Player Max.
Two player, Min and Max, and a half-player, Nature, move a token on a digraph, alternating moves in a cyclic way:

- If the current state \( i \) belongs to Player Min, this player chooses and arc \( i \to j \), and receives \( A_{ji} \) from Player Max.
- The current state \( j \) now belongs to the half-player Nature, Nature throws a dice and next state becomes \( r \) with probability \( P_{jr} \).
- The current state \( r \) now belongs Player Max, this player chooses an arc \( r \to s \), and receives \( B_{rs} \) from Player Max.
- The current state \( s \) now belongs to Player Min, and so on.
If *Min/Max* play $k$ turns according to strategies $\sigma, \tau$, the payment of the game starting from state $i \in [n] := \{\text{Min states}\}$ is denoted by $R^k_i(\sigma, \tau)$. 

$v^k_i$ is the value of the game in horizon $k$, starting from state $i$, and $\sigma^*, \tau^*$ are optimal strategies if $E R^k_i(\sigma^*, \tau^*) \leq v^k_i = E R^k_i(\sigma^*, \tau)$, $\forall \sigma, \tau$.

Theorem (Shapley) $v^k_i = \min_j \{ -A_{ji} + \sum_{r \in \text{Max states}} P_{jr} \max_{s \in \text{Min states}} \left( B_{rs} + v^{k-1}_s \right) \}$, $v^0_i \equiv 0$.

$F: R^n \rightarrow R^n$ Shapley operator $F(x) = (-A^\top) \otimes \min, + (P \times (B \otimes \max, + x)) = A^\# \circ P \circ B(x)$. 

Allamigeon, Gaubert, Katz, Skomra (Inria-X) Nonarchimedean SDP  January 24, 2019, Birmingham  20 / 58
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\textbf{Theorem (Shapley)}

$$v^k_i = \min_{j \in \text{Nature states}} (-A_{ji} + \sum_{r \in \text{Max states}} P_{jr} \max_{s \in \text{Min states}} (B_{rs} + v^{k-1}_s)) \quad , \quad v^0 \equiv 0$$

$$v^k = F(v^{k-1}), \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{Shapley operator}$$
If Min/Max play $k$ turns according to strategies $\sigma, \tau$, the payment of the game starting from state $i \in [n] := \{\text{Min states}\}$ is denoted by $R_i^k(\sigma, \tau)$.

$v^k_i$ is the value of the game in horizon $k$, starting from state $i$, and $\sigma^*, \tau^*$ are optimal strategies if

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The mean payoff vector

\[ \bar{v} := \lim_{k \to \infty} v^k/k = \lim_{k \to \infty} F^k(0)/k \in \mathbb{R}^n \]

does exist and it is achieved by positional stationnary strategies (coro of Kohlberg 1980).
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Mean payoff games: compute the mean payoff vector
The mean payoff vector

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Mean payoff games: compute the mean payoff vector

We say that the mean payoff game with initial state $i$ is (weakly) winning for Max if $\lim_{k} v^k_i / k \geq 0$.

Gurvich, Karzanov and Khachyan asked in 1988 whether the determinisitic version is in P. Still open. Their argument implies membership in $\text{NP} \cap \text{coNP}$, see also Zwick, Paterson. Same is true in the stochastic case (Condon).
Collatz-Wielandt property / winning certificates

\[ T := \mathbb{R} \cup \{-\infty\}, \]

**Theorem (Akian, SG, Guterman IJAC 2912, coro of Nussbaum)**

\[
\max_{i \in n} \bar{v}_i = \overline{cw}(R)
\]

\[
\overline{cw}(F) := \max \{ \lambda \in \mathbb{R} \mid \exists x \in T^n, x \not\equiv -\infty : \lambda e + x \leq F(x) \}
\]

**Corollary**

*Player Max has at least one winning state (i.e., 0 \leq \max_i \bar{v}_i) iff*

\[
\exists x \in T^n, x \not\equiv -\infty, \quad x \leq F(x)
\]
Definition
A square matrix is called a **Metzler matrix** if its off-diagonal entries are nonpositive.
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We suppose $Q^{(1)}, \ldots, Q^{(n)} \in \mathbb{K}^{m \times m}$ are Metzler — the general case will reduce to this one.

Want to decide whether

$$Q(x) = x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succeq 0$$

for some $x \in \mathbb{K}^n_{\geq 0}$, $x \neq 0$. 
If $Q \succeq 0$ is a $m \times m$ symmetric matrix, then, the $1 \times 1$ and $2 \times 2$ principal minors of $Q$ are nonnegative: $Q_{ii} \geq 0$, $Q_{ii}Q_{jj} \geq Q_{ij}^2$. 

Is there a "converse"?

Lemma

Assume that $Q_{ii} \geq 0$, $Q_{ii}Q_{jj} \geq (m-1)^2Q_{ij}^2$. Then $Q \succeq 0$.

Proof.
Can assume that $Q_{ii} \equiv 1$ (consider $\text{diag}(Q) - 1/2Q_{ii} \text{diag}(Q) - 1/2$).

Then, $|Q_{ij}| \leq 1/(m-1)$, and so $Q_{ii} \geq \sum_{j \neq i} |Q_{ij}|$ implies $Q \succeq 0$.

Archimedean modification of Yu's theorem, that the image by the nonarchimedean valuation of the SDP cone is given by $1 \times 1$ and $2 \times 2$ minor conditions.
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Archimedean modification of Yu’s theorem, that the image by the nonarchimedean valuation of the SDP cone is given by $1 \times 1$ and $2 \times 2$ minor conditions.
Let $\mathcal{S} := \{x \in \mathbb{K}^n_{\geq 0} : Q(x) \succeq 0\}$
Let $\mathcal{S} := \{x \in K_n^k : Q(x) \succeq 0\}$

Let $\mathcal{S}^{\text{out}}$ be defined by the $1 \times 1$ and $2 \times 2$ principal minor conditions

$$Q_{ii}(x) \geq 0, \quad Q_{ii}(x)Q_{jj}(x) \geq (Q_{ij}(x))^2$$
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Theorem (Allamigeon, SG, Skomra)

$$\mathcal{S}^{\text{in}} \subset \mathcal{S} \subset \mathcal{S}^{\text{out}}$$

and if $Q$ is tropically generic (valuations of coeffs are generic),

$$\text{val}(\mathcal{S}^{\text{in}}) = \text{val}(\mathcal{S}) = \text{val}(\mathcal{S}^{\text{out}}).$$
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We show that if $X = \cap_k \{x \mid P_k(x) \leq 0\}$, then $\text{val} X = \cap_k \text{val}\{x \mid P_k(x) \leq 0\}$ if the polynomials $P_k$ are tropically generic.
Let $S := \{ x \in K^n_{\geq 0} : Q(x) \succeq 0 \}$
Let $S^{out}$ be defined by the $1 \times 1$ and $2 \times 2$ principal minor conditions
\[ Q_{ii}(x) \geq 0, \quad Q_{ii}(x)Q_{jj}(x) \geq (Q_{ij}(x))^2 \]
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Theorem (Allamigeon, SG, Skomra)

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We show that if $X = \bigcap_k \{ x \mid P_k(x) \leq 0 \}$, then
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generic (apply semi-algebraic version of Kapranov theorem)
Can we describe combinatorially \( \text{val } S \)?
Suppose $Q_{ii}(x) \geq 0$, write $Q_{ii} = Q_{ii}^+ - Q_{ii}^-$. 
Suppose $Q_{ii}(x) \geq 0$, write $Q_{ii} = Q_{ii}^+ - Q_{ii}^-$. Then

$$\text{val } Q_{ii}^+(x) \geq \text{val } Q_{ii}^-(x).$$
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$$Q_{ii}^+(x)Q_{jj}^+(x) + Q_{ii}^-(x)Q_{jj}^-(x) \geq Q_{ii}^+(x)Q_{jj}^-(x) + Q_{ii}^-(x)Q_{jj}^+(x) + (Q_{ij}(x))^2$$
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and so

$$\text{val } Q_{ii}^+(x) + \text{val } Q_{jj}^+(x) \geq 2 \text{ val } Q_{ij}(x)$$
Tropical Metzler spectrahedra

Theorem (tropical Metzler spectrahedra)

For tropically generic Metzler matrices $(Q^{(k)})_k$ the set $\text{val}(S)$ is described by the tropical minor inequalities of order 1 and 2,

\[
\forall i, \max_{Q^{(k)}_{ii} > 0} (x_k + \text{val}(Q^{(k)}_{ii})) \geq \max_{Q^{(l)}_{jj} < 0} (x_l + \text{val}(Q^{(l)}_{jj}))
\]

and

\[
\forall i \neq j, \max_{Q^{(k)}_{ii} > 0} (x_k + \text{val}(Q^{(k)}_{ii})) + \max_{Q^{(k)}_{jj} > 0} (x_k + \text{val}(Q^{(k)}_{jj})) \geq 2 \max_{Q^{(l)}_{ij} < 0} (x_l + \text{val}(Q^{(l)}_{ij})).
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\]

Extends the characterization of \(\text{val}(SDPCONE)\) by Yu.
Lemma

The set \( \text{val}(\mathcal{S}) \) can be equivalently defined as the set of all \( x \) such that for all \( k \) we have

\[
x_k \leq \min_{Q_{ij}^{(k)} < 0} \left( -\text{val}(Q_{ij}^{(k)}) + \frac{1}{2} \left( \max_{Q_{ii}^{(l)} > 0} \left( \text{val}(Q_{ii}^{(l)}) + x_l \right) \right) + \max_{Q_{jj}^{(l)} > 0} \left( \text{val}(Q_{jj}^{(l)}) + x_l \right) \right) \]

In other words, we have

\[
\text{val}(\mathcal{S}) = \{ x \in (\mathbb{R} \cup \{-\infty\})^n : x \leq F(x) \},
\]

where \( F \) is a Shapley operator of a stochastic mean payoff game. We denote this game by \( \Gamma \).
\[ x_k \leq \min_{Q_{ij}^{(k)} < 0} \left( - \text{val}(Q_{ij}^{(k)}) + \frac{1}{2} \left( \max_{Q_{ii}^{(l)} > 0} (\text{val}(Q_{ii}^{(l)}) + x_l) \right. \right. \right. \\
\left. \left. \left. + \max_{Q_{jj}^{(l)} > 0} (\text{val}(Q_{jj}^{(l)}) + x_l) \right) \right) \right) . \]
Reading the Game on the Shapley Operator

\[ x_k \leq \min_{Q_{ij}^{(k)} < 0} \left( -\text{val}(Q_{ij}^{(k)}) + \frac{1}{2} \left( \max_{Q_{ii}^{(l)} > 0} (\text{val}(Q_{ii}^{(l)}) + x_l) \right. \right. \\
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MIN wants to show infeasibility, MAX feasibility
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MIN wants to show infeasibility, MAX feasibility

- state of MIN, \( x_k, 1 \leq k \leq n \)
MIN wants to show infeasibility, MAX feasibility

- state of MIN, $x_k$, $1 \leq k \leq n$
- MIN chooses $\{i, j\}$, $1 \leq i \neq j \leq m$ or $\{i\}$ with $Q_{ii}^{(k)} < 0$, MAX pays to MIN $\text{val}(Q_{ij}^{(k)})$
Reading the Game on the Shapley Operator

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- NATURE throws a dice to decide whether \( i \) or \( j \) is the next state
\[ x_k \leq \min_{Q_{ij}^{(k)} < 0} \left( -\text{val}(Q_{ij}^{(k)}) + \frac{1}{2} \left( \max_{Q_{ii}^{(l)} > 0} \left( \text{val}(Q_{ii}^{(l)}) + x_l \right) \right) + \max_{Q_{jj}^{(l)} > 0} \left( \text{val}(Q_{jj}^{(l)}) + x_l \right) \right) . \]

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- suppose next state of MAX, \( i, 1 \leq i \leq m, \)
Reading the Game on the Shapley Operator

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MIN wants to show infeasibility, MAX feasibility

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- NATURE throws a dice to decide whether \( i \) or \( j \) is the next state
- suppose next state of MAX, \( i, 1 \leq i \leq m \),
- MAX moves to \( x_l \) such that \( Q_{ii}^{(l)} > 0 \), MIN pays to MAX \( \text{val}(Q_{ii}^{(l)}) \).
Main example revisited

\[ Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix}, \]

\[ Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}. \]

Construction of \( \Gamma \)

We construct \( \Gamma \) as follows:
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Construction of \( \Gamma \)

The number of matrices (here: 3) defines the number of states controlled by Player Min.
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Construction of \( \Gamma \)

The size of matrices (here: 3 \( \times \) 3) defines the number of states controlled by Player Max (here: 3).
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Construction of \( \Gamma \)

If \( Q^{(k)}_{ii} \) is negative, then Player Min can move from state \( k \) to state \( i \). After this move Player Max receives \(- \text{val}(Q^{(k)}_{ii})\).
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Construction of \( \Gamma \)

If \( Q_{ji}^{(k)} \) is positive, then Player Max can move from state \( i \) to state \( k \). After this move Player Max receives \( \text{val}(Q_{ji}^{(k)}) \).
Main example revisited

\[ Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

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\[ Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}. \]

Construction of $\Gamma$

If $Q_{ij}^{(k)}$ is nonzero, $i \neq j$, then Player Min have a coin-toss move from state $k$ to states $(i, j)$ and Player Max receives $-\text{val}(Q_{ij}^{(k)})$. 
Main example revisited

\[ Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix}, \]

\[ Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}. \]

Construction of \( \Gamma \)

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Example

There is only one pair of optimal policies

\[ 3 \rightarrow \{1, 3\}, \]
\[ 2 \rightarrow 1. \]
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The value equals \(\frac{3}{40} > 0\).
Example

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\[ \begin{align*}
3 & \rightarrow \{1, 3\} , \\
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\end{align*} \]

The value equals \( \frac{3}{40} > 0 \).

Corollary

The spectrahedral cone \( S \) has a nontrivial point in the positive orthant \( \mathbb{K}^3_{\geq 0} \).
Example

The Shapley operator is given by

\[ F(x) = \left( \frac{x_1 + x_3}{2}, x_1 - 1, \frac{x_2 + x_3}{2} + \frac{7}{8} \right) \]

and \( u = (1.06, 0.02, 1.13) \) is a bias vector, \( F(u) = \lambda e + u, \lambda = \) value

Corollary

The spectrahedral cone \( S \) has a nontrivial point in the positive orthant \( \mathbb{K}^3_{>0} \). For example, it contains the point \( (t^{1.06}, t^{0.02}, t^{1.13}) \).
Helton-Nie conjectured that every convex semialgebraic set is the projection of a spectrahedron.
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Scheiderer (SIAGA, 2018) showed that the cone of nonnegative forms of degree $2d$ in $n$ variables is not representable in this way unless $2d = 2$ or $n \leq 2$ or $(n, 2d) = (3, 4)$, disproving the conjecture. His result implies the conjecture is also false over $\mathbb{K}$. However...
Tropical analogue of Helton-Nie conjecture, cont.

Theorem (Allamigeon, Gaubert, and Skomra, MEGA2017+JSC.)

Fix a set $S \subset \mathbb{R}^n$. TFAE

- $S$ is the image by val of a convex semialgebraic set of $K^n_{>0}$

- $S$ is tropically convex, closed and semilinear

There exists a stochastic game with Shapley operator $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $S = \{ x \in \mathbb{R}^n | x \leq F(x) \}$

There exists a stochastic game with transition probabilities $0, 1/2, 1$ and Shapley operator $F: \mathbb{R}^p \rightarrow \mathbb{R}^p$, with $p \geq n$, such that $S = \text{proj} \{ x \in \mathbb{R}^p | x \leq F(x) \}$
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Tropical analogue of Helton-Nie conjecture, cont.

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- There exists a stochastic game with transition probabilities $0, \frac{1}{2}, 1$ and Shapley operator $F : \mathbb{R}^p \rightarrow \mathbb{R}^p$, with $p \geq n$, such that $S = \text{proj}\{x \in \mathbb{R}^p \mid x \leq F(x)\}$
How to solve the game in practice

- Gurvich, Karzanov and Khachyan pumping algorithm (1988)
  iterative algorithm with hard (discontinuous) thresholds,
  generalized to the stochastic case by Boros, Elbassioni, Gurvich
  and Makino (2015, hard complexity estimates)
How to solve the game in practice


- More refined value type iteration, special case of simple stochastic games **Ibsen-Jensen, Miltersen (2012)**
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Basic value iteration

\[ t x := \max_i x_i \text{ (read “top”)}, \quad b x := \min_i x_i \text{ (read “bot”) \quad} \]

1: procedure \textbf{ValueIteration}(F)

2: \hspace{1em} \triangleright F a Shapley operator from \( \mathbb{R}^n \) to \( \mathbb{R}^n \)

3: \hspace{1em} \triangleright The algorithm will report whether Player Max or Player Min wins the mean payoff game represented by F

4: \hspace{1em} u := 0 \in \mathbb{R}^n

5: \hspace{1em} \textbf{while} t(u) > 0 \text{ and } b(u) < 0 \hspace{1em} \textbf{do} \hspace{1em} u := F(u) \hspace{1em} \triangleright \text{At iteration } \ell, \hspace{1em} u = F^\ell(0) \text{ is the value vector of the game in finite horizon } \ell

6: \hspace{1em} \textbf{done}

7: \hspace{1em} \textbf{if} t(u) \leq 0 \hspace{1em} \textbf{then return} \text{ “Player Min wins”}

8: \hspace{1em} \textbf{else return} \text{ “Player Max wins”}

9: \hspace{1em} \textbf{end}

10: \hspace{1em} \textbf{end}

This is what we implemented to solve the benchmarks of large scale nonarchimedean SDP.
Complexity analysis?
Complexity analysis? Answer: Metric geometry tool
Funk, Hilbert and Thompson metric

$C$ closed convex pointed cone, $x \leq y$ if $y - x \in C$, Funk reverse metric (Papadopoulos, Troyanov):

$$RFunk(x, y) := \log \inf \{ \lambda > 0 | \lambda x \succeq y \}$$
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$C = \mathbb{R}^n_+$, $RFunk(x, y) = \log \max_i y_i / x_i$ (tropical sesquilinear form)
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$C = S_n^+ = \text{positive semidefinite matrices,}$
$RFunk(x, y) = \log \max \text{spec}(x^{-1}y)$. 

Lemma $F$: $\text{int } C \rightarrow \text{int } C$ is order preserving and homogeneous of degree 1 iff $RFunk(F(x), F(y)) \leq RFunk(x, y), \forall x, y \in \text{int } C$. 

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Lemma

$F : \text{int } C \rightarrow \text{int } C$ is order preserving and homogeneous of degree 1 iff

$$RFunk(F(x), F(y)) \leq RFunk(x, y), \quad \forall x, y \in \text{int } C.$$
We can symmetrize Funk’s metric in two ways

\[ d_T(x, y) = \max(\text{RFunk}(x, y), \text{RFunk}(y, x)) \quad \text{Thompsons’ part metric} \]

\[ d_H(x, y) := \text{RFunk}(x, y) + \text{RFunk}(y, x) \quad \text{Hilbert’s projective metric} \]

(plays the role of Euclidean metric in tropical convexity Cohen, SG, Quadrat 2004)

\[ d_H(x, y) = \| \log x - \log y \|_H \quad \text{where} \quad \| z \|_H := \max_{i \in [n]} z_i - \min_{i \in [n]} z_i. \]
A ball in Hilbert’s projective metric is classically and tropically convex.
\[ S(F) := \{ x \in \mathbb{T}^n : x \leq F(x) \}, \quad \mathbb{T} := \mathbb{R} \cup \{ -\infty \} \]

\[ \overline{cw}(F) = \max_i \bar{v}_i, \quad \underline{cw}(F) = \min_i \bar{v}_i \]

(best and worst mean payoffs).
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We say that \( u \in \mathbb{R}^n \) is a bias (tropical eigenvector) if

\[ F(u) = \lambda e + u \]

Then, \( \lambda = \overline{cw}(F) = \overline{cw}(F) \), denoted by \( \rho(F) \) for “spectral radius”, it is unique.
\[ S(F) := \{ x \in T^n : x \leq F(x) \}, \quad T := \mathbb{R} \cup \{-\infty\} \]

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Existence of \( u \) guaranteed by ergodicity conditions, Akian, SG, Hochart, DCSD A.
Definition

An order-preserving and additively homogeneous self-map $F$ of $\mathbb{T}^n$ is said to be *diagonal free* when $F_i(x)$ is independent of $x_i$ for all $i \in [n]$. 

Theorem

Let $F$ be a diagonal free self-map of $\mathbb{T}^n$. Then, $S(F)$ contains a Hilbert ball of positive radius if and only if $cw(F) > 0$. Moreover, when $S(F)$ contains a Hilbert ball of positive radius, the supremum of the radii of the Hilbert balls contained in $S(F)$ coincides with $cw(F)$.
Definition

An order-preserving and additively homogeneous self-map \( F \) of \( \mathbb{T}^n \) is said to be \textit{diagonal free} when \( F_i(x) \) is independent of \( x_i \) for all \( i \in [n] \).

Theorem

Let \( F \) be a diagonal free self-map of \( \mathbb{T}^n \). Then, \( S(F) \) contains a Hilbert ball of positive radius if and only if \( \text{cw}(F) > 0 \). Moreover, when \( S(F) \) contains a Hilbert ball of positive radius, the supremum of the radii of the Hilbert balls contained in \( S(F) \) coincides with \( \text{cw}(F) \).
Extends a theorem of Sergeev, showing that the tropical eigenvalue of $A$ gives the inner radius of the polytropes $\{x \mid x \succeq Ax\}$. 
Extends a theorem of Sergeev, showing that the tropical eigenvalue of $A$ gives the inner radius of the polytropes $\{x \mid x \geq Ax\}$. 
\[ C := \{ x \in K^n : Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \text{ is PSD} \} \]

\[ F : \mathbb{T}^n \to \mathbb{T}^n \text{ Shapley operator of } C. \]

\[ \mathcal{P}(F) : \text{does there exist } x \in \mathbb{T}^n \text{ such that } x \not\equiv -\infty \text{ and } x \leq F(x)? \]
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\[ \mathcal{P}_\mathbb{R}(F) : \text{does there exist } x \in \mathbb{R}^n \text{ such that } x \ll F(x)? \]

**Theorem (Allamigeon, SG, Skomra)**

1. **if** \( \mathcal{P}(F) \) **is infeasible, or equivalently,** \( S(F) \) **is trivial, then** \( C \) **is trivial.**

2. **if** \( \mathcal{P}_\mathbb{R}(F) \) **is feasible, or equivalently,** \( S(F) \) **is strictly nontrivial, then** \( C \) **is strictly nontrivial, meaning that there exists** \( x \in \mathbb{K}_{>0}^n \) **such that the matrix** \( x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \) **is positive definite.**
We define the condition number $\text{cond}(F)$ of the above problem $\mathcal{P}(F)$ by:

$$\left(\inf\{\|u\|_\infty : u \in \mathbb{R}^n, \mathcal{P}(u + F) \text{ is infeasible}\}\right)^{-1}$$ (1)

if $\mathcal{P}(F)$ is feasible, and

$$\left(\inf\{\|u\|_\infty : u \in \mathbb{R}^n, \mathcal{P}(u + F) \text{ is feasible}\}\right)^{-1}$$ (2)

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\[
(\inf\{\|u\|_\infty : u \in \mathbb{R}^n, \mathcal{P}(u + F) \text{ is feasible}\})^{-1}
\]  

(2)

if $\mathcal{P}(F)$ is infeasible.

$u + F$: Shapley operator of a game in which in state $i$, Max receives an additional payment of $u_i$. 
cond_R(F) is defined as cond(F), considering \( P_R(F) \).

**Proposition**

Let \( F \) be a continuous, order-preserving, and additively homogeneous self-map of \( \mathbb{T}^n \). Then,

\[
\text{cond}_R(F) = |cw(F)|^{-1} \quad \text{and} \quad \text{cond}(F) = |\overline{cw}(F)|^{-1}.
\]
\[ R(F) := \inf \{ \|u\|_H : u \in \mathbb{R}^n, F(u) = \rho(F) + u \} . \]

If \( F \) is assumed to have a bias vector \( v \in \mathbb{R}^n \), i.e. \( F(v) = \rho(F) + v \),

\[ |\rho(F)|^{-1} = |cw(F)|^{-1} = |cw(F)|^{-1} = \text{cond}_{\mathbb{R}}(F) = \text{cond}(F). \]

**Theorem (Allamigeon, SG, Katz, Skomra)**

Suppose that the Shapley operator \( F \) has a bias vector and that \( \rho(F) \neq 0 \). Then ValueIteration terminates after

\[ N_{vi} \leq R(F) \text{cond}(F) \]

iterations and returns the correct answer.

Compare with \( \log(R/r) \) in the ellipsoid / interior point methods.
\[ F = A^\# \circ B \circ P \] \hfill (3)

where \( A \in \mathbb{T}^{m \times n} \), \( B \in \mathbb{T}^{m \times q} \), integer entries, \( P \in \mathbb{R}^{q \times n} \) row-stochastic

\[ W := \max \{|A_{ij} - B_{ih}| : A_{ij} \neq -\infty, B_{ih} \neq -\infty, i \in [m], j \in [n], h \in [q]|\]. \]

Probabilities \( P_{il} \) rational with a common denominator \( M \in \mathbb{N}_{>0} \),
\[ P_{il} = Q_{il}/M, \] where \( Q_{il} \in [M] \) for all \( i \in [q] \) and \( l \in [n] \).
A state \( i \in [q] \) is non-deterministic if there are at least two indices \( l, l' \in [n] \) such that \( P_{il} > 0 \) and \( P_{il'} > 0 \).
Theorem

Let $F$ be a Shapley operator as above, still supposing that $F$ has a bias vector and that $\rho(F)$ is nonzero. If $k$ is the number of nondeterministic states of the game, then $\text{cond}(F) \leq nM^{\min\{k,n-1\}}$. 
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Relies on an estimate of Skomra of denominators of invariant measures, obtained from Tutte matrix tree theorem, improves Boros, Elbassioni, Gurvich and Makino.
Theorem (Allamigeon, SG, Katz, Skomra)

\[ R(F) \leq 10n^2 WM^{\min\{k,n-1\}}. \]

We construct a bias by vanishing discount, which yields of the bound on \( R(F) \).

Corollary

Let \( F \) be the above Shapley operator, still supposing that it has a bias vector and that \( \rho(F) \) is nonzero. Then, procedure \textsc{ValueIteration} stops after

\[ N_{vi} \leq 10n^3 WM^2 \min\{k,n-1\} \] (4)

iterations and correctly decides which of the two players is winning.
In the deterministic case, we recover Zwick-Paterson bound.

**Corollary**

Let $F = A^\# \circ B$ be the Shapley operator of a deterministic game, where the finite entries of $A, B \in \mathbb{T}^{m \times n}$ are integers. If there exists $v \in \mathbb{R}^n$ such that $F(v) = \rho(F) + v$ with $\rho(F) \neq 0$, then

$$N_{vi} \leq 2n^2 W.$$
The assumption $\rho(F) \neq 0$ can be relaxed, by appealing to the following perturbation and scaling argument. This leads to a bound in which the exponents of $M$ and of $n$ are increased.

**Corollary**

Let $\mu := nM^{\min\{k, n-1\}}$. Then, procedure `VALUE ITERATION`, applied to the perturbed and rescaled Shapley operator $1 + 2\mu F$, satisfies

$$N_{vi} \leq 21n^4WM^3\min\{k, n-1\}$$

iterations, and this holds unconditionally. If the algorithm reports that Max wins, then Max is winning in the original mean payoff game. If the algorithm reports that Min wins, then Min is strictly winning in the original mean payoff game.

The algorithm can be also adapted to work in finite precision arithmetic.
Tropical homotopy

The condition number controls the critical temperature \( t_c^{-1} \) such that for \( t > t_c \), the archimedean SDP feasibility problem and tropical SDP feasibility problem have the same answer.

\[
\delta(t) := \max_{Q_{ij}^{(k)} \neq 0} \left| |Q_{ij}^{(k)}| - \log_t |Q_{ij}^{(k)}(t)| \right|.
\]

Theorem

Let \( m \geq 2 \), and \( v \) be the value of the stochastic mean payoff game associated with \( Q^{(1)}, \ldots, Q^{(n)} \). Let \( \lambda := \max_k v_k \), and suppose that \( \lambda \neq 0 \). Take any \( t \) such that \( \delta(t) < |\lambda| \) and

\[
t > (2(m - 1)n)^{1/(2|\lambda| - 2\delta(t))}.
\]

Then, the spectrahedron \( S(t) \) is nontrivial if and only if \( \lambda \) is positive.
Concluding remarks

- Showed: **stochastic mean payoff games** polynomial time equivalent to **feasibility of nonarchimedean semidefinite programs** with generic valuations.
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- Controls the critical temperature under which the SDP feasibility problem “freezes” in its tropical state.
Thank you!


