Condition numbers in nonarchimedean semidefinite programming ... and what they say about stochastic mean payoff games

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Feasibility semidefinite programmming problem

Definition (spectrahedron)

Given symmetric matrices $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{R}^{m \times m}$, the associated spectrahedron is defined as

$$\mathcal{S} = \{x \in \mathbb{R}^n \colon Q^{(0)} + x_1 Q^{(1)} + \dots + x_n Q^{(n)} \text{ is positive semidefinite}\} .$$

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- The semidefinite feasibility problem (SFDP) consists in deciding whether $\mathcal{S} = \varnothing$.
- The semidefinite programming problem (SDP) consists in minimizing a linear form over $\mathcal{S}$

- SDP can be solved in polynomial time by the ellipsoid or interior point methods in a restricted sense.

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- We obtain ε-approximate solutions. Complexity bounds:

$$\mathrm{Poly}(n, m, \log \varepsilon, \log R, \log r, \dots),$$

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E. de Klerk and F. Vallentin. "On the Turing model complexity of interior point methods for semidefinite programming". In: *SIAM J. Opt.* 26.3 (2016), pp. 1944–1961

D. Henrion, S. Naldi, and M. Safey El Din. "Exact algorithms for linear matrix inequalities". In: *SIAM J. Opt.* 26.4 (2016), pp. 2512–2539

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use some metric geometry ideas

Generalized Puiseux series

- A (formal generalized) Puiseux series is a series of form

$$\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} ,$$

where the sequence $(\alpha_i)_i \subset \mathbb{R}$ is strictly decreasing and either finite or unbounded and c_i are real. Includes (generalized) Dirichlet series $\alpha_i = -\log i$, $t = \exp(s)$. Hardy, Riesz 1915

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- The subset of absolutely converging (for t large enough) Puiseux series forms a real closed field, denoted here by $\mathbb{K}$.
- We say that $\boldsymbol{x} \geqslant \boldsymbol{y}$ if $\boldsymbol{x}(t) \geqslant \boldsymbol{y}(t)$ for all t large enough. This is a linear order on $\mathbb{K}$.

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Definition (SDFP over Puiseux series)

Given symmetric matrices $\boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)}$, denote

$$\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \cdots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} .$$

Decide if the following spectrahedron is empty

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Proposition

$\boldsymbol{\mathcal{S}} \neq \emptyset$ iff for all t large enough, the following real spectrahedron is non-empty

$$\mathcal{S}(t) = \{x \in \mathbb{R}^n_{\geqslant 0} \colon Q^{(0)}(t) + x_1 Q^{(1)}(t) + \cdots + x_n Q^{(n)}(t) \textit{ is pos. semidef.}\}$$

Proof. $\mathbb{K}$ is the field of germs of univariate functions definable in a o-minimal structure.

Theorem (Allamigeon, SG, Skomra)

There is a correspondence between nonarchimedean semidefinite programming problems and zero-sum stochastic games with perfect information. If the valuations of the matrices $Q^{(i)}$ *are generic, feasibility holds iff Player Max wins the game.*

X. Allamigeon, S. Gaubert, and M. Skomra. "Solving Generic Nonarchimedean Semidefinite Programs Using Stochastic Game Algorithms". In: *Journal of Symbolic Computation* 85 (2018), pp. 25–54. DOI: 10.1016/j.jsc.2017.07.002. eprint: 1603.06916.

Take the spectrahedral cone

$$Q(\boldsymbol{x}) := \begin{bmatrix} t\boldsymbol{x}_3 & -\boldsymbol{x}_1 & -t^{3/4}\boldsymbol{x}_3 \\ -\boldsymbol{x}_1 & t^{-1}\boldsymbol{x}_1 + t^{-5/4}\boldsymbol{x}_3 - \boldsymbol{x}_2 & -\boldsymbol{x}_3 \\ -t^{3/4}\boldsymbol{x}_3 & -\boldsymbol{x}_3 & t^{9/4}\boldsymbol{x}_2 \end{bmatrix} \succcurlyeq 0 \,.$$

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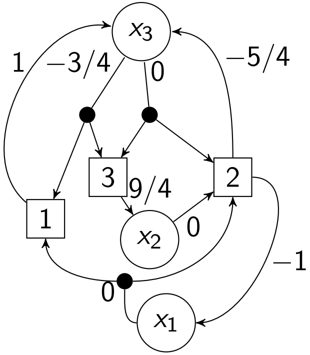
- We associate with $\boldsymbol{Q}(\boldsymbol{x})$ a stochastic game with perfect information.



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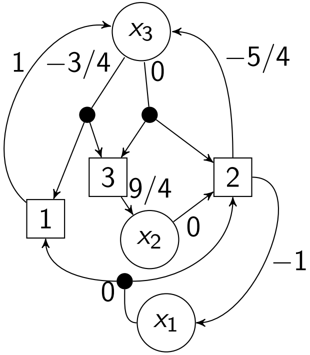

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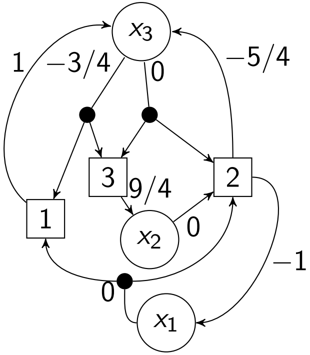

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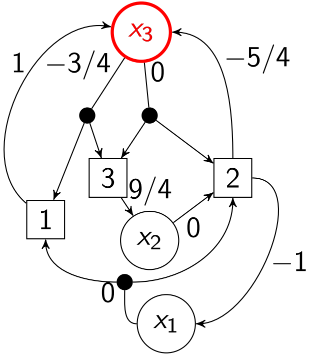

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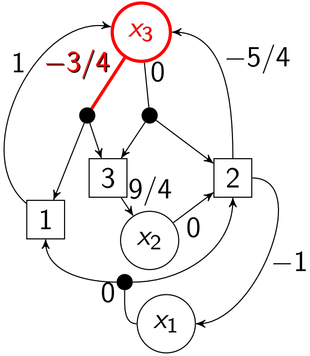

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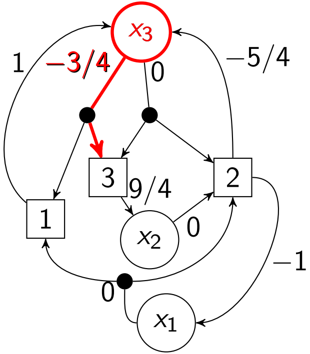

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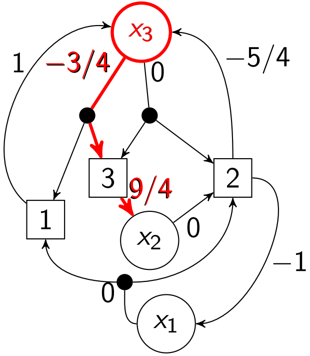

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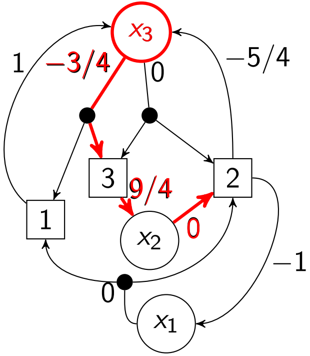

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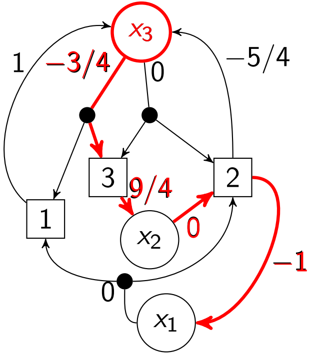

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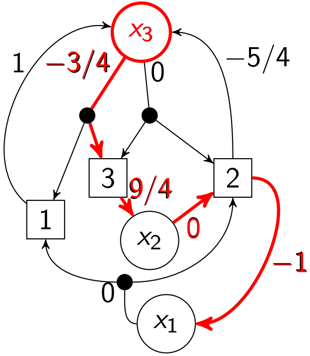
- Max is winning implies that the cone is nontrivial, and yields a feasible point $(t^{1.06}, t^{0.02}, t^{1.13})$.



Benchmark

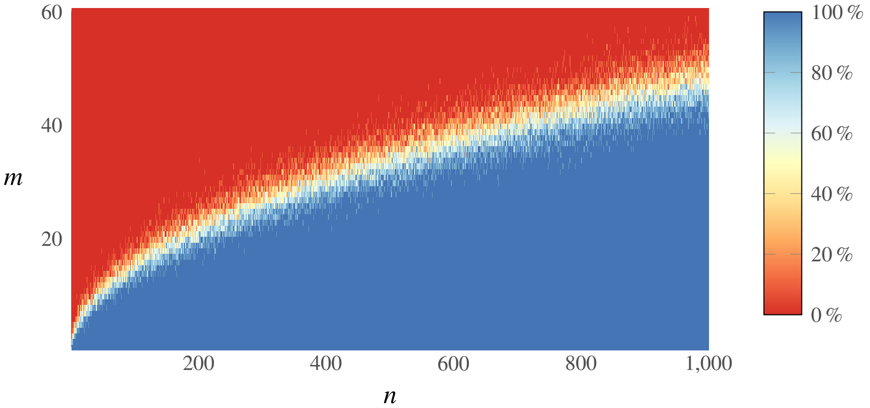
We tested our method on randomly chosen matrices $\boldsymbol{Q}^{(1)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m \times m}$ with positive entries on diagonals and no zero entries. We used the value iteration algorithm.

(n, m)	$(50, 10)$	$(50, 40)$	$(50, 50)$	$(50, 100)$	$(50, 1000)$
time	0.000065	0.000049	0.000077	0.000279	0.026802
(n, m)	$(100, 10)$	$(100, 15)$	$(100, 80)$	$(100, 100)$	$(100, 1000)$
time	0.000025	0.000270	0.000366	0.000656	0.053944
(n, m)	$(1000, 10)$	$(1000, 50)$	$(1000, 100)$	$(1000, 200)$	$(1000, 500)$
time	0.000233	0.073544	0.015305	0.027762	0.148714
(n, m)	$(2000, 10)$	$(2000, 70)$	$(2000, 100)$	$(10000, 150)$	$(10000, 400)$
time	0.000487	1.852221	0.087536	19.919844	2.309174

Table: Execution time (in sec.) of Procedure CheckFeasibility on random instances.

Experimental phase transition for random nonarchimedean SDP

$n = \#$ variables, $m =$ size matrices



The present work on tropical condition numbers grew to explain this picture.

Valuation of Puiseux series

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Lemma

Suppose that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n_{\geqslant 0}$. Then

- $\boldsymbol{x} \geqslant \boldsymbol{y} \implies \operatorname{val}(\boldsymbol{x}) \geqslant \operatorname{val}(\boldsymbol{y})$
- $\operatorname{val}(\boldsymbol{x} + \boldsymbol{y}) = \max(\operatorname{val}(\boldsymbol{x}), \operatorname{val}(\boldsymbol{y}))$
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Thus, val is a morphism from $\mathbb{K}_{\geqslant 0}$ to a semifield of characteristic one, the tropical semifield $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +)$.

Tropical spectrahedra

Definition

Suppose that $\mathcal{S}$ is a spectrahedron in $\mathbb{K}^n_{\geqslant 0}$. Then we say that $\mathrm{val}(\mathcal{S})$ is a tropical spectrahedron.

How can we study these creatures?

A $\mathcal{S} \subset \mathbb{K}^n$ is basic semialgebraic if

$$\mathcal{S} = \{(x_1, \dots, x_n) \in \mathbb{K}^n \colon P_i(x_1, \dots, x_n) \diamond 0, \ \diamond \in \{>, =\}, \forall i \in [q]\}$$

where $P_1, \dots, P_q \in \mathbb{K}[x_1, \dots, x_n]$. A semialgebraic set is a finite union of basic semialgebraic sets.

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A set $S \subset \mathbb{R}^n$ is basic semilinear if it is of the form

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon \ell_i(x_1, \dots, x_n) \diamond h^{(i)}, \ \diamond \in \{>, =\}, \forall i \in [q]\}$$

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Theorem (Alessandrini, Adv. in Geom. 2013)

If $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{>0}$ is semi-algebraic, then $\operatorname{val}(\boldsymbol{\mathcal{S}}) \subset \mathbb{R}^n$ is semilinear and it is closed.

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where $\ell_1, \dots, \ell_q$ are linear forms with integer coefficients, $h^{(1)}, \dots, h^{(q)} \in \mathbb{R}$. A semilinear set is a finite union of basic semilinear sets.

Theorem (Alessandrini, Adv. in Geom. 2013)

If $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{>0}$ is semi-algebraic, then $\operatorname{val}(\boldsymbol{\mathcal{S}}) \subset \mathbb{R}^n$ is semilinear and it is closed.

Constructive version in Allamigeon, SG, Skomra arXiv:1610.06746 using Denef-Pas quantifier elimination in valued fields.

$\mathcal{S} := \mathrm{val}(\boldsymbol{\mathcal{S}})$ is tropically convex

$$\max(\alpha, \beta) = 0,\ u, v \in \mathcal{S} \implies \sup(\alpha e + u, \beta e + v) \in \mathcal{S}\ ,$$

where $e = (1, \dots, 1)^\top$.

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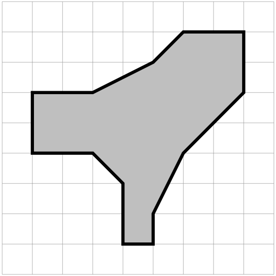

Figure: Tropical spectrahedron.

Theorem (Semi-algebraic version of Kapranov theorem, Allamigeon, SG, Skomra arXiv:1610.06746)

Consider a collection of m regions delimited by hypersurfaces:

$$\boldsymbol{\mathcal{S}}_i := \{x \in \mathbb{K}^n_{\geqslant 0} \mid P_i^-(x) \leqslant P_i^+(x)\}, \qquad i \in [m]$$

where $P_i^{\pm} = \sum_\alpha p_{i,\alpha}^{\pm} x^\alpha \in \mathbb{K}_{\geqslant 0}[x]$, *and let*

$$S_i := \{x \in \mathbb{R}^n \mid \max_\alpha(\operatorname{val} p_{i,\alpha}^- + \langle \alpha, x\rangle) \leqslant \max_\alpha(\operatorname{val} p_{i,\alpha}^+ + \langle \alpha, x\rangle)\}$$

Then

$$\operatorname{val}(\bigcap_{i\in[m]} \boldsymbol{\mathcal{S}}_i) \subset \bigcap_{i\in[m]} \operatorname{val}(\boldsymbol{\mathcal{S}}_i) \subset \bigcap_{i\in[m]} S_i$$

and *the equality holds if* $\bigcap_{i\in[m]} S_i$ *is the closure of its interior; in particular if the valuations* $\operatorname{val} p_{i,\alpha}^{\pm}$ *are generic.*

Example 1.

$$\mathcal{S} = \{x \in \mathbb{K}^3_{>0} \mid x_1^2 \leqslant t x_2 + t^4 x_2 x_3\}$$

$$\operatorname{val} \mathcal{S} = \{x \in \mathbb{R}^3 \mid 2x_1 \leqslant \max(1 + x_2, 4 + x_2 + x_3)\}$$

Example 2.

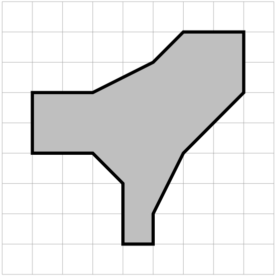

Figure: This set is the closure of its interior.

The correspondence between stochastic mean payoff games and nonarchimedean spectrahedra explained

Stochastic mean payoff games

Two player, Min and Max, and a half-player, Nature, move a token on a digraph, alternating moves in a cyclic way:

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- The current state r now belongs Player Max, this player chosses an arc $r \to s$, and receives B_{rs} from Player Max.
- the current state s now belongs to Player Min, and so on.

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v_i^k is the value of the game in horizon k, starting from state i, and σ^*, τ^* are optimal strategies if

$$\mathbb{E}R_i^k(\sigma^*, \tau) \leqslant v_i^k = \mathbb{E}R_i^k(\sigma^*, \tau^*) \leqslant \mathbb{E}R_i^k(\sigma, \tau^*), \qquad \forall \sigma, \tau$$

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Theorem (Shapley)

$$v_i^k = \min_{j \in \textit{Nature states}} \Big(-A_{ji} + \sum_{r \in \textit{Max states}} P_{jr} \max_{s \in \textit{Min states}} (B_{rs} + v_s^{k-1})\Big) \ , v^0 \equiv 0$$

$$v^k = F(v^{k-1}), \qquad F : \mathbb{R}^n \to \mathbb{R}^n \quad \textit{Shapley operator}$$

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$$F(x) = (-A^\top) \odot_{\min,+} (P \times (B \odot_{\max,+} x)) = A^\sharp \circ P \circ B(x)$$

The mean payoff vector

$$\bar{v} := \lim_{k\to\infty} v^k/k = \lim_{k\to\infty} F^k(0)/k \in \mathbb{R}^n$$

does exist and it is achieved by positional stationnary strategies (coro of Kohlberg 1980).

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Mean payoff games: compute the mean payoff vector

We say that the mean payoff game with initial state i is (weakly) winning for Max if $\lim_k v_i^k/k \geqslant 0$.

Gurvich, Karzanov and Khachyan asked in 1988 whether the determinisitic version is in P. Still open. Their argument implies membership in NP $\cap$ coNP, see also Zwick, Paterson. Same is true in the stochastic case (Condon).

Collatz-Wielandt property / winning certificates

$\mathbb{T} := \mathbb{R} \cup \{-\infty\}$,

Theorem (Akian, SG, Guterman IJAC 2912, coro of Nussbaum)

$$\max_{i \in n} \bar{v}_i = \overline{\mathrm{cw}}(R)$$

$$\overline{\mathrm{cw}}(F) := \max\left\{\lambda \in \mathbb{R} \mid \exists x \in \mathbb{T}^n, x \not\equiv -\infty : \lambda e + x \leqslant F(x)\right\}$$

Corollary

Player Max has at least one winning state (i.e., $0 \leqslant \max_i \bar{v}_i$) iff

$$\exists x \in \mathbb{T}^n, x \not\equiv -\infty, \qquad x \leqslant F(x)$$

Definition

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Want to decide whether

$$Q(x) = x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succcurlyeq 0$$

for some $x \in \mathbb{K}^n_{\geqslant 0}$, $x \neq 0$.

If $Q \succcurlyeq 0$ is a $m \times m$ symmetric matrix, then, the 1×1 and 2×2 principal minors of Q are nonnegative: $Q_{ii} \geqslant 0$, $Q_{ii}Q_{jj} \geqslant Q_{ij}^2$.

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Assume that $Q_{ii} \geqslant 0$, $Q_{ii}Q_{jj} \geqslant (m-1)^2 Q_{ij}^2$. *Then* $Q \succcurlyeq 0$.

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Archimedean modification of Yu's theorem, that the image by the nonarchimedean valuation of the SDP cone is given by 1×1 and 2×2 minor conditions.

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Theorem (Allamigeon, SG, Skomra)

$$\mathcal{S}^{in} \subseteq \mathcal{S} \subseteq \mathcal{S}^{out}$$

and if Q is tropically generic (valuations of coeffs are generic),

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$\text{val}\, X = \cap_k \text{val}\{x \mid P_k(x) \leqslant 0\}$ if the polynomials P_k are tropically generic (apply semi-algebraic version of Kapranov theorem)

Can we describe combinatorially val $\mathcal{S}$?

Suppose $\boldsymbol{Q}_{ii}(\boldsymbol{x}) \geqslant 0$, write $\boldsymbol{Q}_{ii} = \boldsymbol{Q}_{ii}^{+} - \boldsymbol{Q}_{ii}^{-}$.

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and so

$$\operatorname{val} Q_{ii}^{+}(x) + \operatorname{val} Q_{jj}^{+}(x) \geqslant 2 \operatorname{val} Q_{ij}(x)$$

Tropical Metzler spectrahedra

Theorem (tropical Metzler spectrahedra)

For tropically generic Metzler matrices $(Q^{(k)})_k$ *the set* $\mathrm{val}(\mathcal{S})$ *is described by the tropical minor inequalities of order* 1 *and* 2,

$$\forall i, \max_{Q^{(k)}_{ii} > 0} (x_k + \mathrm{val}(Q^{(k)}_{ii})) \geqslant \max_{Q^{(l)}_{jj} < 0} (x_l + \mathrm{val}(Q^{(l)}_{jj}))$$

and

$$\forall i \neq j, \max_{Q^{(k)}_{ii} > 0} (x_k + \mathrm{val}(Q^{(k)}_{ii})) + \max_{Q^{(k)}_{jj} > 0} (x_k + \mathrm{val}(Q^{(k)}_{jj})) \geqslant 2 \max_{Q^{(l)}_{ij} < 0} (x_l + \mathrm{val}(Q^{(l)}_{ij})) \, .$$

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and

$$\forall i \neq j,\ \max_{Q_{ii}^{(k)}>0} (x_k + \mathrm{val}(Q_{ii}^{(k)})) + \max_{Q_{jj}^{(k)}>0} (x_k + \mathrm{val}(Q_{jj}^{(k)})) \geqslant 2 \max_{Q_{ij}^{(l)}<0} (x_l + \mathrm{val}(Q_{ij}^{(l)})) \,.$$

Extends the characterization of $\mathrm{val}(SDPCONE)$ by Yu. .

From spectrahedra to Shapley operators

Lemma

The set $\operatorname{val}(\boldsymbol{\mathcal{S}})$ *can be equivalently defined as the set of all* x *such that for all* k *we have*

$$x_k \leqslant \min_{\boldsymbol{Q}_{ij}^{(k)}<0} \Big(-\operatorname{val}(\boldsymbol{Q}_{ij}^{(k)}) + \frac{1}{2}\big(\max_{\boldsymbol{Q}_{ii}^{(l)}>0} (\operatorname{val}(\boldsymbol{Q}_{ii}^{(l)}) + x_l) + \max_{\boldsymbol{Q}_{jj}^{(l)}>0} (\operatorname{val}(\boldsymbol{Q}_{jj}^{(l)}) + x_l)\big)\Big) .$$

In other words, we have

$$\operatorname{val}(\boldsymbol{\mathcal{S}}) = \{x \in (\mathbb{R} \cup \{-\infty\})^n \colon x \leqslant F(x)\} ,$$

where F *is a Shapley operator of a stochastic mean payoff game. We denote this game by* Γ*.*

Reading the Game on the Shapley Operator

$$x_k \leqslant \min_{Q_{ij}^{(k)}<0} \Big(-\operatorname{val}(Q_{ij}^{(k)}) + \frac{1}{2}\big(\max_{Q_{ii}^{(l)}>0} (\operatorname{val}(Q_{ii}^{(l)}) + x_l) + \max_{Q_{jj}^{(l)}>0} (\operatorname{val}(Q_{jj}^{(l)}) + x_l)\big)\Big) \,.$$

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MIN wants to show infeasibility, MAX feasibility

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- state of MIN, x_k, $1 \leqslant k \leqslant n$
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Reading the Game on the Shapley Operator

$$x_k \leqslant \min_{Q_{ij}^{(k)}<0} \Big(-\operatorname{val}(Q_{ij}^{(k)}) + \frac{1}{2}\big(\max_{Q_{ii}^{(l)}>0} (\operatorname{val}(Q_{ii}^{(l)}) + x_l) + \max_{Q_{jj}^{(l)}>0} (\operatorname{val}(Q_{jj}^{(l)}) + x_l)\big)\Big) .$$

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Reading the Game on the Shapley Operator

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- state of MIN, x_k, $1 \leqslant k \leqslant n$
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- NATURE throws a dice to decide whether i or j is the next state
- suppose next state of MAX, i, $1 \leqslant i \leqslant m$,
- MAX moves to x_l such that $Q_{ii}^{(l)} > 0$, MIN pays to MAX $\operatorname{val} Q_{ii}^{(l)}$.

Main example revisited

$$Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix},$$

$$Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}.$$

Construction of Γ

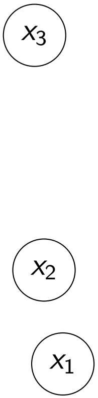
We construct Γ as follows:

Main example revisited

$$Q^{(1)} := \begin{bmatrix} 0 & -1 & 0 \\ -1 & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q^{(2)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & t^{9/4} \end{bmatrix},$$

$$Q^{(3)} := \begin{bmatrix} t & 0 & -t^{3/4} \\ 0 & t^{-5/4} & -1 \\ -t^{3/4} & -1 & 0 \end{bmatrix}.$$



Construction of Γ

The number of matrices (here: 3) defines the number of states controlled by Player Min.

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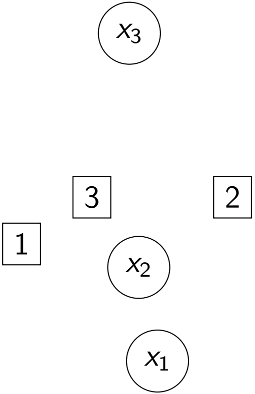

Construction of Γ

The size of matrices (here: 3×3) defines the number of states controlled by Player Max (here: 3).

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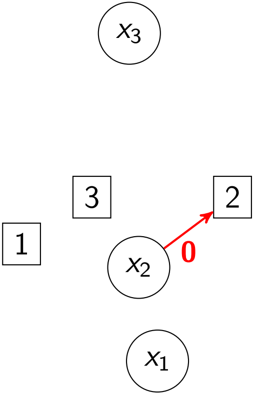

Construction of Γ

If $Q_{ii}^{(k)}$ is negative, then Player Min can move from state k to state i. After this move Player Max receives $-\operatorname{val}(Q_{ii}^{(k)})$.

Main example revisited

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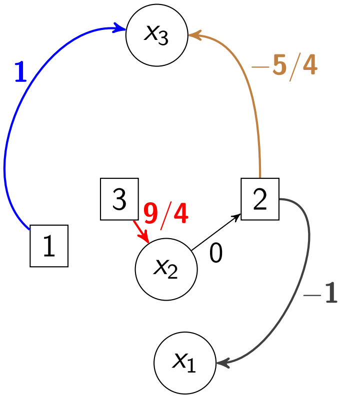

Construction of Γ

If $Q^{(k)}_{ii}$ is positive, then Player Max can move from state i to state k. After this move Player Max receives $\mathrm{val}(Q^{(k)}_{ii})$.

Main example revisited

$$Q^{(1)} := \begin{bmatrix} 0 & -\mathbf{1} & 0 \\ -\mathbf{1} & t^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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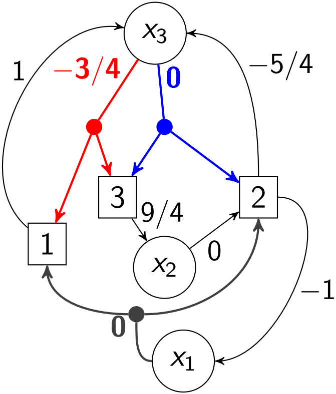

Construction of Γ

If $Q^{(k)}_{ij}$ is nonzero, $i \neq j$, then Player Min have a coin-toss move from state k to states (i, j) and Player Max receives $-\operatorname{val}(Q^{(k)}_{ij})$.

Main example revisited

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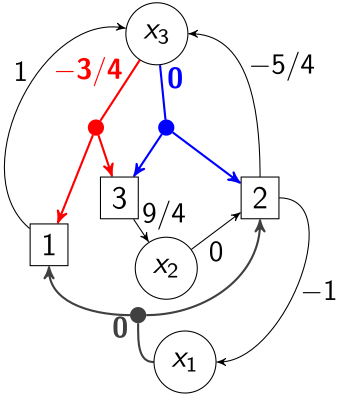

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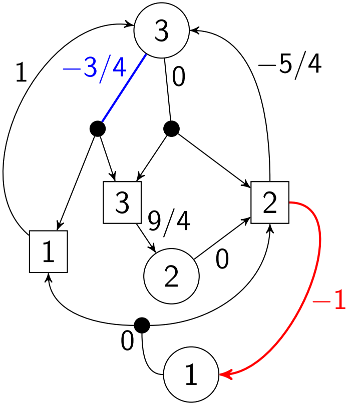
Example

There is only one pair of optimal policies

$$\textcircled{3} \to \left\{ \boxed{1}, \boxed{3} \right\},$$

$$\boxed{2} \to \textcircled{1}.$$



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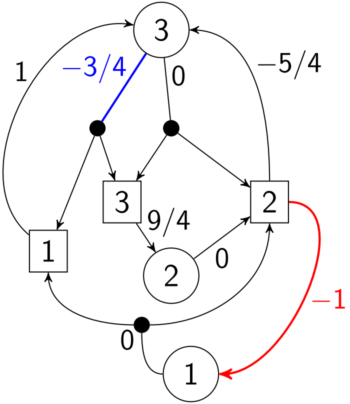
The value equals $3/40 > 0$.



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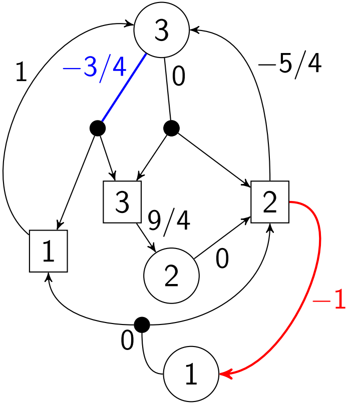


Corollary

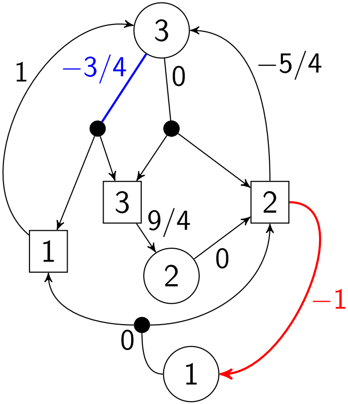
The spectrahedral cone $\mathcal{S}$ has a nontrivial point in the positive orthant $\mathbb{K}^3_{\geqslant 0}$.

Example

The Shapley operator is given by

$$F(x) = (\frac{x_1 + x_3}{2}, x_1 - 1, \frac{x_2 + x_3}{2} + \frac{7}{8})$$

and $u = (1.06, 0.02, 1.13)$ is a bias vector, $F(u) = \lambda e + u$, $\lambda =$ value



Corollary

The spectrahedral cone $\mathcal{S}$ has a nontrivial point in the positive orthant $\mathbb{K}^3_{\geqslant 0}$. For example, it contains the point $(t^{1.06}, t^{0.02}, t^{1.13})$.

Tropical analogue of Helton-Nie conjecture

Helton-Nie conjectured that every convex semialgebraic set is the projection of a spectrahedron.

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Scheiderer (SIAGA, 2018) showed that the cone of nonnegative forms of degree $2d$ in n variables is not representable in this way unless $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$, disproving the conjecture. His result implies the conjecture is also false over $\mathbb{K}$. However…

Tropical analogue of Helton-Nie conjecture, cont.

Theorem (Allamigeon, Gaubert, and Skomra, MEGA2017+JSC.)

Fix a set $\mathcal{S} \subset \mathbb{R}^n$*. TFAE*

- $\mathcal{S}$ *is the image by* val *of a convex semialgebraic set of* $\mathbb{K}^n_{>0}$

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- *There exists a stochastic game with transition probabilities $0, \frac{1}{2}, 1$ and Shapley operator $F : \mathbb{R}^p \to \mathbb{R}^p$, with $p \geqslant n$, such that $\mathcal{S} = \text{proj}\{x \in \mathbb{R}^p \mid x \leqslant F(x)\}$*

How to solve the game in practice

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- value iteration, Zwick Paterson (1996) in the deterministic case.
- more refined value type iteration, special case of simple stochastic games Ibsen-Jensen, Miltersen (2012)

Basic value iteration

$\mathbf{t}x := \max_i x_i$ (read "top"), $\mathbf{b}x := \min_i x_i$ (read "bot")

procedure VALUEITERATION(F)
    ▷ *F a Shapley operator from $\mathbb{R}^n$ to $\mathbb{R}^n$*
    ▷ *The algorithm will report whether Player Max or Player Min wins the mean payoff game represented by F*
    $u := 0 \in \mathbb{R}^n$
    while $\mathbf{t}(u) > 0$ and $\mathbf{b}(u) < 0$ **do** $u := F(u)$ ▷ *At iteration ℓ, $u = F^\ell(0)$ is the value vector of the game in finite horizon ℓ*
    done
    if $\mathbf{t}(u) \leqslant 0$ **then return** "Player Min wins"
    else return "Player Max wins"
    end
end

This is what we implemented to solve the benchmarks of large scale nonarchimedean SDP.

Complexity analysis?

Complexity analysis? Answer: Metric geometry tool

Funk, Hilbert and Thompson metric

C closed convex pointed cone, $x \leqslant y$ if $y - x \in C$, Funk reverse metric (Papadopoulos, Troyanov):

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$C = S_n^+ =$ positive semidefinite matrices,
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Lemma

F : int $C \to$ int C is order preserving and homogeneous of degree 1 iff

$$\mathrm{RFunk}(F(x), F(y)) \leqslant \mathrm{RFunk}(x, y), \qquad \forall x, y \in \mathrm{int}\, C \; .$$

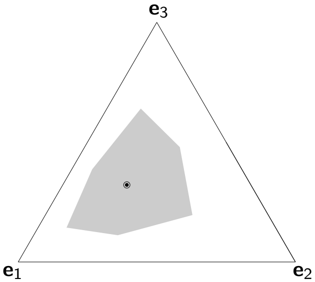
We can symmetrize Funk's metric in two ways

$d_T(x,y) = \max(\mathrm{RFunk}(x,y), \mathrm{RFunk}(y,x))$ Thompsons' part metric

$d_H(x,y) := \mathrm{RFunk}(x,y) + \mathrm{RFunk}(y,x)$ Hilbert's projective metric

(plays the role of Euclidean metric in tropical convexity Cohen, SG, Quadrat 2004)

$d_H(x,y) = \|\log x - \log y\|_H$ where $\|z\|_H := \max_{i\in[n]} z_i - \min_{i\in[n]} z_i$.



A ball in Hilbert's projective metric is classically and tropically convex.

$$\mathcal{S}(F) := \{x \in \mathbb{T}^n \colon x \leqslant F(x)\}, \qquad \mathbb{T} := \mathbb{R} \cup \{-\infty\}$$

$$\overline{\mathrm{cw}}(F) = \max_i \bar{v}_i, \qquad \underline{\mathrm{cw}}(F) = \min_i \bar{v}_i$$

(best and worst mean payoffs).

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We say that $u \in \mathbb{R}^n$ is a *bias* (tropical eigenvector) if

$$F(u) = \lambda e + u$$

Then, $\lambda = \underline{\mathrm{cw}}(F) = \overline{\mathrm{cw}}(F)$, denoted by $\rho(F)$ for "spectral radius", it is unique.

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Existence of u guaranteed by *ergodicity conditions*, Akian, SG, Hochart, DCSD A.

Definition

An order-preserving and additively homogeneous self-map F of $\mathbb{T}^n$ is said to be *diagonal free* when $F_i(x)$ is independent of x_i for all $i \in [n]$.

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Theorem

Let F be a diagonal free self-map of $\mathbb{T}^n$. Then, $\mathcal{S}(F)$ contains a Hilbert ball of positive radius if and only if $\underline{\mathrm{cw}}(F) > 0$. Moreover, when $\mathcal{S}(F)$ contains a Hilbert ball of positive radius, the supremum of the radii of the Hilbert balls contained in $\mathcal{S}(F)$ coincides with $\underline{\mathrm{cw}}(F)$.

Biggest Hilbert ball in a tropical polyhedra

Extends a theorem of Sergeev, showing that the tropical eigenvalue of A gives the inner radius of the polytropes $\{x \mid x \geqslant Ax\}$.

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$$\mathcal{C} := \left\{ \boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \cdots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)} \text{ is PSD} \right\}$$

$F\colon \mathbb{T}^n \to \mathbb{T}^n$ Shapley operator of $\mathcal{C}$.

$\mathscr{P}(F)$: does there exist $x \in \mathbb{T}^n$ such that $x \not\equiv -\infty$ and $x \leqslant F(x)$?

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$\mathscr{P}_{\mathbb{R}}(F)$: does there exist $x \in \mathbb{R}^n$ such that $x \ll F(x)$?

Theorem (Allamigeon, SG, Skomra)

(i) *if $\mathscr{P}(F)$ is infeasible, or equivalently, $\mathcal{S}(F)$ is trivial, then $\mathcal{C}$ is trivial.*

(ii) *if $\mathscr{P}_{\mathbb{R}}(F)$ is feasible, or equivalently, $\mathcal{S}(F)$ is strictly nontrivial, then $\mathcal{C}$ is strictly nontrivial, meaning that there exists $\boldsymbol{x} \in \mathbb{K}^n_{>0}$ such that the matrix $\boldsymbol{x}_1 \boldsymbol{Q}^{(1)} + \cdots + \boldsymbol{x}_n \boldsymbol{Q}^{(n)}$ is positive definite.*

We define the *condition number* $\mathrm{cond}(F)$ of the above problem $\mathscr{P}(F)$ by:

$$(\inf\{\|u\|_\infty \colon u \in \mathbb{R}^n\,,\ \mathscr{P}(u+F) \text{ is infeasible}\})^{-1} \tag{1}$$

if $\mathscr{P}(F)$ is feasible, and

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if $\mathscr{P}(F)$ is infeasible.

$u+F$: Shapley operator of a game in which in state i, Max receives an additional payment of u_i.

$\mathrm{cond}_{\mathbb{R}}(F)$ is defined as $\mathrm{cond}(F)$, considering $\mathscr{P}_{\mathbb{R}}(F)$.

Proposition

Let F be a continuous, order-preserving, and additively homogeneous self-map of $\mathbb{T}^n$. Then,

$$\mathrm{cond}_{\mathbb{R}}(F) = |\underline{\mathrm{cw}}(F)|^{-1} \textit{ and } \mathrm{cond}(F) = |\overline{\mathrm{cw}}(F)|^{-1}.$$

$$R(F) := \inf\left\{\|u\|_{\mathrm{H}} \colon u \in \mathbb{R}^n,\ F(u) = \rho(F) + u\right\} .$$

If F is assumed to have a bias vector $v \in \mathbb{R}^n$, i.e. $F(v) = \rho(F) + v$,

$$|\rho(F)|^{-1} = |\underline{\mathrm{cw}}(F)|^{-1} = |\overline{\mathrm{cw}}(F)|^{-1} = \mathrm{cond}_{\mathbb{R}}(F) = \mathrm{cond}(F) .$$

Theorem (Allamigeon, SG, Katz, Skomra)

Suppose that the Shapley operator F has a bias vector and that $\rho(F) \neq 0$. Then ValueIteration *terminates after*

$$N_{\mathrm{vi}} \leqslant R(F)\,\mathrm{cond}(F)$$

iterations and returns the correct answer.

Compare with $\log(R/r)$ in the ellipsoid / interior point methods.

$$F = A^\sharp \circ B \circ P \tag{3}$$

where $A \in \mathbb{T}^{m\times n}$, $B \in \mathbb{T}^{m\times q}$, integer entries, $P \in \mathbb{R}^{q\times n}$ row-stochastic

$W := \max\{|A_{ij} - B_{ih}| \colon A_{ij} \neq -\infty,\ B_{ih} \neq -\infty,\ i \in [m], j \in [n],\ h \in [q]\}$

Probabilities P_{il} rational with a common denominator $M \in \mathbb{N}_{>0}$, $P_{il} = Q_{il}/M$, where $Q_{il} \in [M]$ for all $i \in [q]$ and $l \in [n]$.
A state $i \in [q]$ is *nondeterministic* if there are at least two indices $l, l' \in [n]$ such that $P_{il} > 0$ and $P_{il'} > 0$.

Theorem

Let F be a Shapley operator as above, still supposing that F has a bias vector and that $\rho(F)$ is nonzero. If k is the number of nondeterministic states of the game, then $\operatorname{cond}(F) \leqslant nM^{\min\{k,n-1\}}$.

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Relies on an estimate of Skomra of denominators of invariant measures, obtained from Tutte matrix tree theorem, improves Boros, Elbassioni, Gurvich and Makino

Theorem (Allamigeon, SG, Katz, Skomra)

$$R(F) \leqslant 10n^2 W M^{\min\{k,n-1\}} .$$

We construct a bias by vanishing discount, which yields of the bound on $R(F)$.

Corollary

Let F be the above Shapley operator, still supposing that it has a bias vector and that $\rho(F)$ is nonzero. Then, procedure ValueIteration *stops after*

$$N_{\mathrm{vi}} \leqslant 10n^3 W M^{2\min\{k,n-1\}} \tag{4}$$

iterations and correctly decides which of the two players is winning.

In the deterministic case, we recover Zwick-Paterson bound.

Corollary

Let $F = A^\sharp \circ B$ be the Shapley operator of a deterministic game, where the finite entries of $A, B \in \mathbb{T}^{m\times n}$ are integers. If there exists $v \in \mathbb{R}^n$ such that $F(v) = \rho(F) + v$ with $\rho(F) \neq 0$, then

$$N_{\rm vi} \leqslant 2n^2 W \enspace .$$

The assumption $\rho(F) \neq 0$ can be relaxed, by appealing to the following perturbation and scaling argument. This leads to a bound in which the exponents of M and of n are increased.

Corollary

Let $\mu := nM^{\min\{k,n-1\}}$. Then, procedure VALUEITERATION, *applied to the perturbed and rescaled Shapley operator $1 + 2\mu F$, satisfies*

$$N_{\mathrm{vi}} \leqslant 21 n^4 W M^{3\min\{k,n-1\}}$$

iterations, and this holds unconditionally. If the algorithm reports that Max wins, then Max is winning in the original mean payoff game. If the algorithm reports that Min wins, then Min is strictly winning in the original mean payoff game.

The algorithm can be also adapted to work in finite precision arithmetic.

Tropical homotopy

The condition number controls the critical temperature t_c^{-1} such that for $t > t_c$, the archimedean SDP feasibility problem and tropical SDP feasibility problem have the same answer.

$$\delta(t) := \max_{Q_{ij}^{(k)} \neq \mathbb{0}} \left| |Q_{ij}^{(k)}| - \log_t |\boldsymbol{Q}_{ij}^{(k)}(t)| \right| .$$

Theorem

Let $m \geq 2$, and v be the value of the stochastic mean payoff game associated with $Q^{(1)}, \dots, Q^{(n)}$. Let $\lambda := \max_k v_k$, and suppose that $\lambda \neq 0$. Take any t such that $\delta(t) < |\lambda|$ and

$$t > (2(m-1)n)^{1/(2|\lambda| - 2\delta(t))} .$$

Then, the spectrahedron $\boldsymbol{\mathcal{S}}(t)$ is nontrivial if and only if λ is positive.

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Thank you !

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