Basic definitions and concepts
Optimal assignments with supervisions
Tropical Jacobi identity

Optimal assignments with supervisions

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BASIC DEFINITIONS AND CONCEPTS
Tropical linear algebra

- Consider real numbers $\mathbb{R} \cup \{-\infty\}$ equipped with
  \[ a \circ b = a + b, \quad a \oplus b := \max(a, b). \]

- Semifield with $0 = -\infty$, $1 = 0$.
  I.e. $a^{-1} = -a$ and $\not\exists \ominus a$.

- Applies to matrices and vectors entry-wise:
  \[
  (A \oplus B)_{i,j} := (A_{i,j} \oplus B_{i,j})
  \\
  (A \circ B)_{i,j} := \bigoplus_{k} A_{i,k} \circ B_{kj}
  \]
Correspondence: \( I, J \) minor of \( A^{-1} \) to \( J^c, I^c \) minor of \( A \).

Theorem (the classical identity)

For \( A \in \text{GL}_n(\mathbb{F}) \), \( I, J \subseteq [n] \) s.t. \( |I| = |J| = k \)

\[
(DA^{-1}D)_{I,J}^{\wedge k} = (\det(A))^{-1}A_{J^c,I^c}^{\wedge n-k},
\]

where \( D_{i,i} = (-1)^i \) and \( D_{i,j} = 0 \) for \( i \neq j \).

Theorem (the tropical identity)

Let $M \in \mathbb{R}^{n \times n}_{\max}$ and $I, J \subseteq [n]$ s.t. $|I| = |J| = k$.

Either:

$$[D(\det(M)^{-1}\text{adj}(M))D]_{I,J}^k = \det(M)^{-1}M_{J^c,I^c}^{\land n-k}$$

Or:

There exist distinct bijections $\pi, \sigma \in S_{I,J}$ such that

$$[\text{adj}(M)]_{I,J}^k = \bigodot_{i \in I} \text{adj}(M)_{i,\pi(i)} = \bigodot_{i \in I} \text{adj}(M)_{i,\sigma(i)}.$$

The tropical determinant is actually the permanent with respect to $\oplus, \odot$. That is

$$
\text{per}(A) = \bigoplus_{\pi \in S_n} \bigotimes_{i \in [n]} A_{i, \pi(i)} = \max_{\pi \in S_n} \sum_{i \in [n]} A_{i, \pi(i)},
$$

Graphically: the permutation of optimal weight in the graph of $A$.
Combinatorially: the 'optimal assignment problem'.

Let $\pi, \tau$ be permutations of identical weight $w$.
* In supertropical $w(\pi) \oplus w(\tau)$ is singular.
* In symmetrized $w(\pi) \oplus w(\tau)$ is singular if $\pi$ and $\tau$ are permutations of opposite signs.
2013 - PhD (with L.Rowen) - Conjecture: Let $A^\nabla = \text{per}^{-1}(A)\text{adj}(A)$ (sort of inverse). Then (supertropically) coefficient-wise
$$\text{per}(A)f_{A^\nabla}(x) = x^n f_A(x^{-1}) \oplus \text{'singular polynomial'}.$$ 

That is, $\bigoplus A_{I_{\text{I}},I}$ corresponds to $\bigoplus A_{I_{\text{c}},I_{\text{c}}}$. 
[Y.Shitov 'On the Char. Polynomial of a Supertropical Adjoint Matrix', LAA.]
How did it form?

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  \[ \text{per}(A)f_{A^\nabla}(x) = x^n f_A(x^{-1}) \oplus \text{‘singular polynomial’}. \]
  That is, $\oplus A^\nabla_I$ corresponds to $\oplus A_{Ic,Jc}$.

  [Y.Shitov 'On the Char. Polynomial of a Supertropical Adjoint Matrix’, LAA.]

- **2015 - Postdoc (with M.Akian and S.Gaubert)** - (symmetrized) Tropical Jacobi:
  \[ [D(\det(M)^{-1}\text{adj}(M))D]_{I,J}^{\wedge k} = \det(M)^{-1}M_{Jc,Ic}^{\wedge n-k} \oplus \text{‘singular matrix’}. \]
  So, entry-wise, for every $I, J$, and including signs.
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  That is, $\oplus A^{\nabla}_{I,I}$ corresponds to $\oplus A_{I^c,I^c}$. [Y. Shitov ‘On the Char. Polynomial of a Supertropical Adjoint Matrix’, LAA.]

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  \[ [D(\det(M))^{-1}\adj(M))D]_{I,J}^{\wedge k} = \det(M)^{-1} M^{\wedge n-k}_{j^c,i^c} \oplus \text{‘singular matrix’}. \]
  So, entry-wise, for every $I, J$, and including signs.

- **2016-2018 (with McCaig and Sergeev) -** Graph theory version:
  Every optimal $(1, k)$-regular multigraph of $M$ w.r.t. $I, J$
  either: corresponds to an optimal bijection w.r.t. $I^c, J^c$, 
or: there exists another optimal $(1, k)$-regular w.r.t. $I, J$. [That is, combinatorially, without signs, which led to the application.]
Definitions: digraphs

- A **weighted digraph** $G$ is a pair $(V_G, E_G)$ where
  - $V_G$ is set of nodes and
  - $E_G \subseteq V_G \times V_G$ is set of directed edges on $|V_G|$ nodes (allowing loops and multiple edges).

- **Weight**: $w(i, j)$ for each $(i, j)$.

- A **bipartite graph** is a triple $(V_{H,1}, V_{H,2}, E_H)$ s.t.
  \[ i \in V_{H,1} \iff j \in V_{H,2} \text{ for every } (i, j) \in E_H, \text{ weighted: } w(i, j) \text{ for each } (i, j). \]
**Associated digraphs**

- **Matrix** $M \in \mathbb{R}_{\text{max}}^{n \times n} \rightarrow$ **weighted digraph** $G_M = (V, E)$, where $V = [n]$ and $E = \{(i, j): M_{i,j} \neq 0\}$, and weight $w(i, j) = M_{i,j}$.

- **Weighted digraph** $G = ([n], E, w) \rightarrow$ **matrix** $M_G$,

  where $$(M_G)_{i,j} = \begin{cases} w(i,j) & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$
Digraphs and matrices

\[ M = M_G = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & 0 & 0 \\ M_{3,1} & M_{3,2} & 0 \end{pmatrix} \]

\[ G = G_M \]

\[ M_{1,1} \quad M_{1,2} \quad M_{1,3} \]

\[ M_{2,1} \quad 0 \quad 0 \]

\[ M_{3,1} \quad M_{3,2} \quad 0 \]

\[ 1 \quad 2 \quad 3 \]
Associated bipartite graphs

- **Matrix** $M \in \mathbb{R}_{\text{max}}^{m \times n}$ \(\longrightarrow\) **bipartite graph** $G_M = (V_{H_1}, V_{H_2}, E_H)$, $|V_{H_1}| = m$, $|V_{H_2}| = n$, and $E_H = \{(i, j) : M_{i,j} \neq -\infty\}$, weight $w(i, j) = M_{i,j}$.

- **Bipartite graph** $G = (V_{H_1}, V_{H_2}, E_H)$ \(\longrightarrow\) **matrix** $M_G \in \mathbb{R}_{\text{max}}^{m \times n}$, $|V_{H_1}| = m$, $|V_{H_2}| = n$

  where \((M_G)_{i,j} = \begin{cases} w(i, j) & \text{if } (i, j) \in E_H, \\ 0 & \text{otherwise.} \end{cases}\)

- **Digraph** $DG = ([n], E_D) \longleftrightarrow \text{bipartite graph } BG = ([2n], E_B)$, s.t. $(i, j + n) \in E_B$ for every $(i, j) \in E_D$. 
Bipartite graphs and matrices

\[ M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & 0 & 0 \\ M_{3,1} & M_{3,2} & 0 \end{pmatrix} \]
Definitions: assignment problems

- Let $S_n$ denote the set of permutations on $[n]$, and $S_{I,J}$ denote the set of bijections from $I \subseteq [n]$ to $J \subseteq [n]$ (that is, $|I| = |J|$).

- For $M \in \mathbb{R}^{n \times n}$ tropical permanent is defined by

$$\text{per}(M) = \max_{\pi \in S_n} \sum_{i \in [n]} M_{i,\pi(i)} = \bigoplus_{\pi \in S_n} \bigotimes_{i \in [n]} M_{i,\pi(i)}.$$

- A permutation $\pi$ of maximal weight in $\text{per}(M)$ is an optimal permutation in $M$ or $G_M$. That is,

$$\text{per}(M) = \bigotimes_{i \in [n]} M_{i,\pi(i)} = \sum_{i \in [n]} w(i, \pi(i)).$$

- This is identical to the set of optimal assignments, i.e., optimal solutions to the assignment problem in the bipartite graph associated with $M$. 

Adi Niv
A non-$0$ tropical "summand" $w(\pi) = \bigodot_{i \in [n]} M_{i, \pi(i)}$ in $\text{per} M$, or in $M \leftrightarrow \text{permutation-subgraph}$ of $G_M$

with $V(E_\pi) = [n]$, $E_\pi = \{(i, \pi(i)) \forall i \in [n]\}$.

$(1 \ 2 \ 4)(5 \ 3)(6)$

(and the same for path, cycle, bijection,...)
A non-0 tropical "summand" $w(\pi) = \bigodot_{i \in [n]} M_{i, \pi(i)}$ in $\text{per} M$ \iff assignment subgraph with
$V(E_{\pi}) = [n] + [n]$, $E_{\pi} = \{(i, \pi(i)) \; \forall i \in [n]\}$.

(2 1 3)

(and the same for path, cycle, bijection,...)
**Basic definitions and concepts**

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**Tropical Jacobi identity**

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**$k$-regular graphs**

- A graph or digraph $G = (V, E)$ is **$k$-regular** if
  \[ \forall v \in V : \deg(v) = k \text{ (if } G \text{ is a graph)} \]
  \[ \forall v \in V : \deg^+(v) = \deg^-(v) = k \text{ (if } G \text{ is a digraph)} \]

- **Observation:** Let $G = ([n], E)$ be a $k$-regular digraph, then

  \[
  E = \biguplus_{i \in [k]} E_{\rho_i}, \quad \rho_i \in S_n
  \]

  i.e., a disjoint union of edge sets of $k$ permutation-subgraphs $G_i = ([n], E_{\rho_i})$ for some $\rho_i$, for $i \in [k]$.

  [Hall’s Marriage Thm and Z.Izhakian and L.Rowen, Supertropical matrix algebra.]

- **So** $G = ([n], \biguplus_{i \in [k]} E_{\rho_i})$. 

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Hall’s Marriage Theorem
Hall’s Marriage Theorem
Let $G$ be $k$-regular (with $\rho_1, \ldots, \rho_k$). We say $G$ is $(1, k)$-regular w.r.t. $I, J$ with $|I| = |J| = k$ if there exist $e_i \in E_{\rho_i} \ \forall i \in [k]$ s.t. $s(e_i) \in I$, $t(e_i) \in J$ and

$$(V(E_\pi), E_\pi = \{e_1, \ldots, e_k\})$$

is a bijection-subgraph.

We denote

$$G = ([n], \bigcup_{i \in [k]} E_{\rho_i}, \pi).$$
Example: (1,3)-regular graph
Example: (1,3)-regular graph
Denote by \( M^\wedge k \in \mathbb{R}_{\text{max}}^{(n \choose k) \times (n \choose k)} \) the tropical \( k^{th} \) compound matrix of \( M \) defined by

\[
M^\wedge k_{I,J} = \bigoplus_{\sigma \in S_{I,J}} \bigodot_{i \in I} M_{i,\sigma(i)} = \max_{\sigma \in S_{I,J}} \sum_{i \in I} M_{i,\sigma(i)}
\]

\( \forall I, J \subseteq [n] : |I| = |J| = k, I, J \) ordered lexicographically.

In particular, \( M^\wedge 1 = M \), \( M^\wedge 0 = 1 \) and \( \text{per}(M) = M^\wedge n \) is the tropical permanent of \( M \).

\( \text{adj}(M)_{i,j} = M^{\wedge n-1}_{\{j\}^c,\{i\}^c} \)
is the \((i,j)\) entry of the tropical adjugate of \( M \).
We say that \( ([n], \bigcup_{i \in [k]} E_{\rho_i}, \sigma) \) is an \textbf{optimal} \((1, k)\)-regular multigraph of \( G \) w.r.t. \( I, J \) if

\[
\left( \sum_{i \in [k]} w(\rho_i) \right) - w(\sigma) \geq \left( \sum_{i \in [k]} w(\rho'_i) \right) - w(\sigma'),
\]

for every \((1, k)\)-regular multigraph \( ([n], \bigcup_{i \in [k]} E_{\rho'_i}, \sigma') \) of \( G \).

\textbf{Equivalently}

\[
(adj(M_G))^{\wedge k}_{j, l} = \bigodot_{i \in I} (adj(M_G))_{\sigma(i), i}, \quad \text{where}
\]

\[
(adj(M_G))_{\sigma(i), i} = \bigodot_{j \in \{i\}^c} (M_G)_{j, \rho_i(j)}.
\]
Example: (1,3)-regular graph

\[ \sum_{i \in [3]} w(\rho_i) \]
Example: (1,3)-regular graph

\[
\left( \sum_{i \in [3]} w(\rho_i) \right), \ w(\sigma)
\]
Example: (1,3)-regular graph

\[
\left( \sum_{i \in [3]} w(\rho_i) \right) - w(\sigma) = (\text{adj}(M_G))_{j,l}^k
\]
OPTIMAL ASSIGNMENTS WITH SUPERVISIONS
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Assignments with supervisions

- **Supervisions:** Let
  - $M \in \mathbb{R}^{n \times n}_{\text{max}}$
  - $\rho_t \in S_n$ for $t \in [k]$ be $k$ assignments,
  - $(i_t, j_t) \in I \times J$ be $k$ edges s.t. $\sigma(i_t) = j_t$ for $\sigma \in S_I, J$.

$\sigma$ defines **supervisions** on $\{\rho_t : t \in [k]\}$ if $\rho_t(i_t) = j_t \ \forall t$.

- The **base value** of these assignments with supervisions is
  \[
  \sum_{t \in [k]} \left( w(\rho_t, M) - M_{i_t, \sigma(i_t)} \right) = \sum_{t=1}^{k} \sum_{i \neq i_t} M_{i, \rho_t(i)}. \]

- This is also the **weight** of $(1, k)$-regular multigraph
  $([n], \bigcup_{t \in [k]} E_{\rho_t, \sigma})$. 
Assignments with supervisions of people \( \{1, 3, 6\} \) on tasks \( \{2, 3, 5\} \)
The optimal base value of $k$ assignments with supervisions $I$ on $J$ is
\[
\bigoplus_{\sigma \in S_{J,I}} w(\sigma, \text{adj}(M)_{J,I}) = [\text{adj}(M)]^{\wedge k}_{J,I}
\]

It is also the weight of an optimal $(1, k)$-regular multigraph w.r.t. $I$ and $J$. 
Example

Let

\[
M = \begin{pmatrix}
0 & 1 & -2 & -4 \\
-3 & 0 & 5 & 2 \\
-5 & 4 & 0 & 6 \\
-1 & -6 & 3 & 0
\end{pmatrix}, \quad \text{then}\ \text{adj}(M) = \begin{pmatrix}
9 & 10 & 6^\bullet & 12 \\
10 & 9 & 5^\bullet & 11 \\
5 & 6 & 2 & 6^\bullet \\
8 & 9 & 5 & 9
\end{pmatrix}.
\]

**Goal:** Find optimal assignments with supervisions of \( I = \{2, 4\} \) on \( J = \{1, 2\} \).

The **maximum base value** is given by

\[
\text{adj}(M)_{J, I}^2 = \text{per} \begin{pmatrix}
10 & 12 \\
9 & 11
\end{pmatrix} = 21^\bullet.
\]

The **optimal bijections (supervisions)** are

\[
\sigma_1 = (2 \rightarrow 1)(4 \rightarrow 2) \quad \text{and} \quad \sigma_2 = (2 \rightarrow 2)(4 \rightarrow 1).
\]
We found that $\sigma_1 : (2 \to 1)(4 \to 2)$ is optimal.

Supervision $2 \to 1$ corresponds to

\[
M_{\{1,3,4\},\{2,3,4\}} = \begin{pmatrix}
1 & -2 & -4 \\
4 & 0 & 6 \\
-6 & 3 & 0
\end{pmatrix}.
\]

$\beta_1 = (1 \to 2)(3 \to 4)(4 \to 3) \in S_{\{1,3,4\},\{2,3,4\}}$, $\rho_1 = (1 \to 2)(2 \to 1)(3 \to 4)(4 \to 3) \in S_4$.

For supervision $4 \to 2$, we similarly obtain:

$\beta_2 = (1 \to 1)(2 \to 3)(3 \to 4) \in S_{\{1,2,3\},\{1,3,4\}}$, $\rho_2 = (1 \to 1)(2 \to 3)(3 \to 4)(4 \to 2) \in S_4$.

**Optimal** $(1, k)$-regular multigraph:

$F = (E_{\rho_1} \cup E_{\rho_2}, \sigma_1)$. 
TROPICAL JACOBI IDENTITY IN GRAPHS
Theorem (Tropical Jacobi identity)

Let $M \in \mathbb{R}^{n \times n}_{\max}$ and $I, J \subseteq [n]$ such that $|I| = |J| = k$. Then:

1. $[\text{per}(M)^{-1}\text{adj}(M)]_{I,J}^{\wedge k} = \text{per}(M)^{-1}M_{I^c,J^c}^{\wedge n-k}$ OR
2. There exist distinct bijections $\pi, \sigma \in S_{I,J}$ such that

$$[\text{adj}(M)]_{I,J}^{\wedge k} = \sum_{i \in I} \text{adj}(M)_{i,\pi(i)} = \sum_{i \in I} \text{adj}(M)_{i,\sigma(i)}.$$
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Tropical adjugate and optimal multigraphs

- \((\text{adj} M)^{\wedge k}_{j, i} = \) the weight of an optimal \((1, k)\)-regular multigraph
  \[ F = ([n], \biguplus_{i \in [k]} E_{\rho_i}, \pi) \text{ w.r.t. } I, J \subseteq [n]. \]

- We will assume that \(M_{i, i} = 1\) and \(\text{Id} \in S_n\) is an optimal assignment in \(M\). That is, \(\text{per}(M) = \bigodot_{i \in [n]} M_{i, i} = 1\).

  Indeed, this normalization \(M \mapsto PM\) process is invertible, so by Binet-Cauchy and classical Jacobi, if tropical Jacobi holds for \(PM\), it holds for \(M\).

- This means \(\text{Id} \in S_k\) is an optimal assignment of weight \(1\) in \(M\) for every \(k\), and in particular, loops are ‘equally or more optimal’ than every cycle.
Case of unicycle permutations $\rho_i$

$\rho_i : \quad \bullet \quad \cdots \quad \bullet$

loops on $[n] \setminus V(C_i)$

$\beta_i : \quad \bullet \quad \cdots \quad \bullet$

loops on $[n] \setminus V(P_i)$

$e_i \in E_\pi$

$s(P_i) = t(e_i)$

$s(e_i) = t(P_i)$

$P_i$
Theorem

- Let $\text{Id}$ be an optimal permutation in $G = ([n], E)$.
- Let $F = ([n], \bigcup_{i \in [k]} E_{\rho_i}, \pi)$ be an optimal $(1, k)$-regular multigraph of $G$ with respect to $I, J \subseteq [n]$.

**EITHER:**

\[ w(F) = w(\sigma) \text{ where } \sigma \in S_{I^c, J^c} \text{ is an optimal bijection,} \]

**OR:**

There exists $\tilde{\pi} \in S_{I, J}$ and $\tau_i \in S_n$ s.t.

\[ F' = ([n], \bigcup_{i \in [k]} E_{\tau_i}, \tilde{\pi}) \neq F \text{ is also an optimal } (1, k)\text{-regular multigraph} \text{ with respect to } I, J. \]
Example

Let

\[
A = \begin{pmatrix}
0 & -1 & -5 & -4 \\
-6 & 0 & -2 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -7 & 0 & 0
\end{pmatrix}.
\]

Then

\[
\text{adj}(A) = \begin{pmatrix}
0 & -1 & -2 & -2 \\
-3 & 0 & -1 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -3 & 0 & 0
\end{pmatrix} \leftrightarrow \frac{\text{or}}{I_4 - A} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix}.
\]
The first case

- **Case 1:** All paths $P_i$ for $i \in [k]$ are pairwise disjoint.

- Under this condition, we take
  \[ \sigma = \text{composition } P_1 \circ \ldots \circ P_k \text{ with disjoint loops.} \]
  That is:
  
  (a) All sources and targets of $P_i$ are disjoint,
  
  (b) Sources and targets are disjoint to all intermediate nodes,
  
  (c) All intermediate nodes of $P_i$ are disjoint.
The first case

(1, k)-regular

\[ \rho_1 = \rho_1 \mid_{s(e_1)} \]

\[ \rho_2 = \rho_2 \mid_{s(e_2)} \]

\[ \vdots \]

\[ \rho_k = \rho_k \mid_{s(e_k)} \]

\[ S_n \]

\[ C_1 \]

\[ C_2 \]

\[ \vdots \]

\[ C_k \]

\[ \rho_1 \mid_{s(e_1)} \]

\[ \rho_2 \mid_{s(e_2)} \]

\[ \vdots \]

\[ \rho_k \mid_{s(e_k)} \]

\[ \operatorname{Id}^{k-1} \]

\[ \cup \]

\[ \pi' \]

\[ \pi \]

\[ S_{l^c, j^c} \]

\[ S_{l, j} \]

\[ \tau_1 = \operatorname{Id} \]

\[ \tau_{k-1} = \operatorname{Id} \]

\[ \tau_k = \pi' \circ \pi \]

Figure: Case (1): Optimal (1, k)-regular multigraph $F$ corresponds to an...
Example: Case 1

\[
A = \begin{pmatrix}
0 & -1 & -5 & -4 \\
-6 & 0 & -2 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -7 & 0 & 0
\end{pmatrix}, \quad \text{adj}(A) = \begin{pmatrix}
0 & -1 & -2 & -2 \\
-3 & 0 & -1 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -3 & 0 & 0
\end{pmatrix}
\]

Case 1 in the theorem:

\[
\text{adj}(A)^3_{\{1\},\{4\}} = -2 = A_{4,1} = A^1_{\{4\},\{1\}}
\]
Example: Case 1

- **Left:** $\text{adj}(A)^{\wedge 3}_{\{2,3,4\},\{1,2,3\}}$ is the weight of $F$ (an optimal $(1, k)$-regular multigraph).
- **Right:** $\sigma$ is the (optimal) bijection: $J^c = \{4\} \rightarrow I^c = \{1\}$. Joined with loops and the supervision edges, it makes a permutation.

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array}
\rightarrow
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array}
\]

$(2) \ (3) \ 4 \rightarrow 1$
Violation of a)

Case 2a): There exists a source which is also a target. In this case \( \exists i, j \in [k] : t(P_j) = s(P_i) \).

\[
t(P_j) = s(P_i)
\]
Violation of a)

Construct $F' = \left( \bigcup_{i \in [k]} E_{\tau_i}, \pi' \right)$ by:

- Replacing $\rho_i, \rho_j \rightarrow (\tau_i = \tau), (\tau_j = \text{Id})$,
- Keeping $\tau_\ell = \rho_\ell$ for all $\ell \neq i, j$,
- $\tilde{\pi}$ is formed from $\pi$ by replacing
  $(t(P_j), s(P_j)), (t(P_i), s(P_i)) \rightarrow (t(P_i), s(P_j)), (t(P_j), s(P_i))$. 
Example: Case 2a

\[
A = \begin{pmatrix}
0 & -1 & -5 & -4 \\
-6 & 0 & -2 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -7 & 0 & 0
\end{pmatrix}, \quad \text{adj}(A) = \begin{pmatrix}
0 & -1 & -2 & -2 \\
-3 & 0 & -1 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -3 & 0 & 0
\end{pmatrix}
\]

Case 2a in the theorem For \( I = \{1, 2, 3\} \) and \( J = \{1, 3, 4\} \) we have

\[
\text{adj}(A)_{J,I}^3 = -3^\bullet > A_{I^C,J^C}^1 = A_{4,2} = -7.
\]
Example: Case 2a

- **Left** is attained by two bijections in adj(A):
  
  (3), 4 → 1 → 2 and (1)(3), 4 → 2.

- These bijections represent, in A, the following choices for 3 assignments with supervisions:

  obtained by the same set of reorganized edges:

  
  \[ 4 \rightarrow 1 \quad 1 \rightarrow 2 \quad \text{and} \quad 4 \rightarrow 1 \rightarrow 2 \]
Violation of b)

Case 2b: There exists an intermediate node which is also a source or a target.
Assume w.l.o.g. that Case 2a does not occur.
Violation of b)

Construct $F' = (\bigcup_{i \in [k]} E_{\tau_i}, \pi')$ by:

- Replacing $\rho_i, \rho_j \rightarrow (\tau_i = \tau), (\tau_j = \tau')$,
- Keeping $\tau_\ell = \rho_\ell$ for all $\ell \neq i, j$,
- $\tilde{\pi}$ is formed from $\pi$ by replacing
  $(t(P_j), s(P_j)), (t(P_i), s(P_i)) \rightarrow (t(P_i), s(P_j)), (t(P_j), s(P_i))$. 
Example: Case 2b

\[ A = \begin{pmatrix}
0 & -1 & -5 & -4 \\
-6 & 0 & -2 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -7 & 0 & 0 \\
\end{pmatrix}, \quad \text{adj}(A) = \begin{pmatrix}
0 & -1 & -2 & -2 \\
-3 & 0 & -1 & -1 \\
-3 & -4 & 0 & -3 \\
-2 & -3 & 0 & 0 \\
\end{pmatrix} \]

Case 2b in the theorem For \( I = \{1, 2\} \) and \( J = \{3, 4\} \) we have:

\[ \text{adj}(A)^{\wedge 2}_{J,I} = -6^\bullet = (\text{adj}(A)_{3,1}\text{adj}(A)_{4,2}) \oplus (\text{adj}(A)_{3,2}\text{adj}(A)_{4,1}), \]

\[ A^{\wedge 2}_{I,C,J,C} = -6 = A_{3,2}A_{4,1}. \]
Example: Case 2b

- In this case $\text{adj}(A)^{\wedge 2}_{ji}$ is attained twice AND equality holds in the tropical Jacobi identity.

- There are three sets of 2 assignments obtaining the optimal base value:

  
  \begin{align*}
  &3 \to 1 \quad 4 \to 1 \to 2 \quad \text{and} \quad 4 \to 1 \quad 3 \to 1 \to 2 \\
  \end{align*}

  The first two are obtained by the same set of reorganized edges:

  - $3 \to 1 \quad 4 \to 1 \to 2$
  - $4 \to 1 \quad 3 \to 1 \to 2$

  The third is case1 - disjoint paths: $3 \to 2 \quad 4 \to 1$ obtaining $A^{\wedge 2}_{i^C,j^C}$. 

Adi Niv
Violation of c)

Case 2c): There exists an intermediate node common to two paths. Assume w.l.o.g. that Cases 2a, 2b do not occur.

\[ s(P_j) \xrightarrow{t} s(P_j) \xrightarrow{t} t(P_i) = s(P_j) \xrightarrow{t} s(P_j) \xrightarrow{t} t(P_i) \]

composed with loops

\[ s(P_j) \xrightarrow{\tau'} t(P_i) \]

\[ s(P_i) \xrightarrow{\tau} t(P_j) \]
Construct \( F' = (\bigcup_{i \in [k]} E_{\tau_i}, \pi') \) by:

- Replacing \( \rho_i, \rho_j \rightarrow (\tau_i = \tau), (\tau_j = \tau') \),
- Keeping \( \tau_\ell = \rho_\ell \) for all \( \ell \neq i, j \),
- \( \tilde{\pi} \) is formed from \( \pi \) by replacing 
  \( (t(P_j), s(P_j)), (t(P_i), s(P_i)) \rightarrow (t(P_i), s(P_j)), (t(P_j), s(P_i)) \).
Example: Case 2C

(4) (5) 1 → 2 → 3 , (1) (3) 4 → 2 → 5

with the bijection 3 → 1 , 5 → 4, becomes

(3) (4) 1 → 2 → 5 , (1) (5) 4 → 2 → 3

with the bijection 5 → 1 , 3 → 4:
Supervised assignment optimization

Monday      Tuesday      Wednesday      Thursday
1. Work schedule  2. Lunch  3. Tips
Basic definitions and concepts
Optimal assignments with supervisions
Tropical Jacobi identity

Supervised assignment optimization

Monday  Tuesday  Wednesday  Thursday
1. Work schedule (Monday)  2. Lunch  3. Tips (Wednesday)
4. Carpool  5. Inventory (Tuesday)  6. Leftovers (Thursday)
Supervised assignment optimization

Monday  
1. Work schedule (Monday)

Tuesday  
2. Lunch

Wednesday  
3. Tips (Wednesday)
4. Carpool
5. Inventory (Tuesday)

Thursday  
6. Leftovers (Thursday)
Supervised assignment optimization

\[(1) \ 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \in S_{\{1, 2, 3, 4\}, \{1, 3, 4, 5\}}\]
Basic definitions and concepts
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Supervised assignment optimization

\[(1)(2)(3)(4)(5)(6) \quad (1 \ 2 \ 3 \ 4 \ 5 \ 6)) \quad (2 \ 3)(4 \ 5)(1 \ 6) \quad (2 \ 3 \ 4)(1 \ 6 \ 5)\]

\[(1) \ 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \in S_{\{1,2,3,4\},\{1,3,4,5\}}\]
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\[(1)(2)(3)(4)(5)(6) \quad (1 \ 2 \ 3 \ 4 \ 5 \ 6) \quad (2 \ 3)(4 \ 5)(1 \ 6) \quad (2 \ 3 \ 4)(1 \ 6 \ 5)\]

\[(1) \ 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \in S_{\{1,2,3,4\},\{1,3,4,5\}}\]

\[(6) \ 5 \rightarrow 2 \quad \text{or} \quad 5 \rightarrow 6 \rightarrow 2 \in S_{\{5,6\},\{2,6\}}\]
Supervised assignment optimization

\[(6) \ 5 \rightarrow 2 \quad \text{or} \quad 5 \rightarrow 6 \rightarrow 2 \in S\{5,6\},\{2,6\}\]
THANK YOU!