Drawing Flow Notebook

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History

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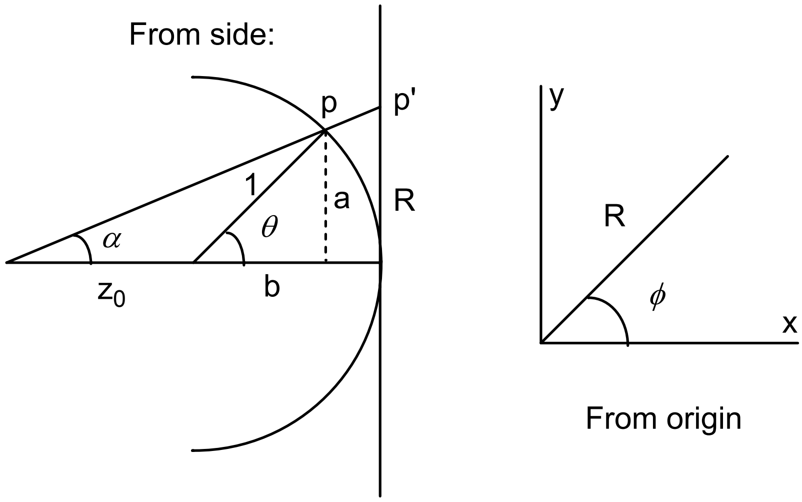
Introduction

There are 2 approaches to drawing flow—by polar projection onto a 2D image plane or, more amibitiously, as a 3D vector plot using Mathematika.

Polar projection



$$\text{From the side view:}$$

$$a = \sin \vartheta$$

$$b = \cos \vartheta$$

$$\text{so } \tan \alpha = \frac{a}{z_0 + b} = \frac{\sin \vartheta}{z_0 + \cos \vartheta}$$

$$\text{but } \tan \alpha = \frac{R}{z_0 + 1}$$

$$\text{so } \frac{\sin \vartheta}{z_0 + \cos \vartheta} = \frac{R}{z_0 + 1}$$

$$\text{hence } R = \frac{\sin \vartheta (z_0 + 1)}{z_0 + \cos \vartheta}$$

$$\text{And, using the view from origin:}$$

$$\left. \begin{aligned} x &= R \cos \varphi \\ y &= R \sin \varphi \end{aligned} \right\} \quad (1)$$

Note that we can use this general formula to approximate parallel projection, simply by setting z_0—the "projection distance"—to a large value. In practice, values of 2 or 3 seem to give good polar results.

3D plotting using Mathematika

Unfortunately, Mathematika doesn't seem to have a decent 3D vector plotting routine that works with spherical coordinates. However, it does have one that works with Cartesian coordinates, so all you have to do is to convert spherical flow back into Cartesian flow.

By definition, in the optic array, $\rho = 1$, so, using the standard spherical coordinate formulae (see Coordinate Conversions notebook):

$$\left.\begin{aligned} x &= \sin\vartheta\cos\varphi \\ y &= \sin\vartheta\sin\varphi \\ z &= \cos\vartheta \end{aligned}\right\} \quad (2)$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial x}{\partial \vartheta}\frac{d\vartheta}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} \quad (3)$$

By definition, $\frac{d\rho}{dt} = 0$, so

$$\frac{\partial x}{\partial \rho}\frac{d\rho}{dt} = 0 \quad (4)$$

From (2), $\frac{\partial}{\partial \vartheta}(\sin\vartheta\cos\varphi) = \cos\vartheta\cos\varphi$, so

$$\frac{\partial x}{\partial \vartheta}\frac{d\vartheta}{dt} = \cos\vartheta\cos\varphi\frac{d\vartheta}{dt} \quad (5)$$

From (2), $\frac{\partial}{\partial \varphi}(\sin\vartheta\cos\varphi) = -\sin\vartheta\sin\varphi$, so

$$\frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = -\sin\vartheta\sin\varphi\frac{d\varphi}{dt} \quad (6)$$

Putting (4), (5) and (6) into (3) gives:

$$\dot{x} = \cos\vartheta\cos\varphi\dot{\vartheta} - \sin\vartheta\sin\varphi\dot{\varphi} \quad (7)$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial y}{\partial \vartheta}\frac{d\vartheta}{dt} + \frac{\partial y}{\partial \varphi}\frac{d\varphi}{dt} \qquad (8)$$

$$\text{By definition, } \frac{d\rho}{dt} = 0, \text{ so}$$

$$\frac{\partial y}{\partial \rho}\frac{d\rho}{dt} = 0 \qquad (9)$$

$$\text{From (2), } \frac{\partial}{\partial \vartheta}(\sin\vartheta \sin\varphi) = \cos\vartheta \sin\varphi, \text{ so}$$

$$\frac{\partial y}{\partial \vartheta}\frac{d\vartheta}{dt} = \cos\vartheta \sin\varphi \frac{d\vartheta}{dt} \qquad (10)$$

$$\text{From (2), } \frac{\partial}{\partial \varphi}(\sin\vartheta \sin\varphi) = \sin\vartheta \cos\varphi, \text{ so}$$

$$\frac{\partial y}{\partial \varphi}\frac{d\varphi}{dt} = \sin\vartheta \cos\varphi \frac{d\varphi}{dt} \qquad (11)$$

$$\text{Putting (9), (10) and (11) into (8) gives:}$$

$$\dot{y} = \cos\vartheta \sin\varphi \dot{\vartheta} + \sin\vartheta \cos\varphi \dot{\varphi} \qquad (12)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial z}{\partial \vartheta}\frac{d\vartheta}{dt} + \frac{\partial z}{\partial \varphi}\frac{d\varphi}{dt} \qquad (13)$$

$$\text{By definition, } \frac{d\rho}{dt} = 0, \text{ so}$$

$$\frac{\partial z}{\partial \rho}\frac{d\rho}{dt} = 0 \qquad (14)$$

$$\text{From (2), } \frac{\partial}{\partial \vartheta}(\cos\vartheta) = -\sin\vartheta, \text{ so}$$

$$\frac{\partial z}{\partial \vartheta}\frac{d\vartheta}{dt} = -\sin\vartheta\varphi \frac{d\vartheta}{dt} \qquad (15)$$

$$\text{From (2), } \frac{\partial}{\partial \varphi}(\cos\vartheta) = 0, \text{ so}$$

$$\frac{\partial y}{\partial \varphi}\frac{d\varphi}{dt} = 0 \qquad (16)$$

$$\text{Putting (14), (15) and (16) into (13) gives:}$$

$$\dot{z} = -\sin\vartheta \, \dot{\vartheta} \qquad (17)$$

Summary

$$\begin{pmatrix} 1 \\ \vartheta \\ \varphi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\vartheta} \\ \dot{\varphi} \end{pmatrix} \Rightarrow \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\vartheta\cos\varphi\dot{\vartheta} - \sin\vartheta\sin\varphi\dot{\varphi} \\ \cos\vartheta\sin\varphi\dot{\vartheta} + \sin\vartheta\cos\varphi\dot{\varphi} \\ -\sin\vartheta\dot{\vartheta} \end{pmatrix} \quad (18)$$

To draw the resulting field, use the Mathematika function ListPlotVectorField3D[], which expects a list of positions and vectors in the following form:

$$\{\{\{x_1, y_1, z_1\}, \{\dot{x}_1, \dot{y}_1, \dot{z}_1\}\}, \{\{x_2, y_2, z_2\}, \{\dot{x}_2, \dot{y}_2, \dot{z}_2\}\}, \cdots, \{\{x_n, y_n, z_n\}, \{\dot{x}_n, \dot{y}_n, \dot{z}_n\}\}\}$$