Simple Mediation in a Cheap-Talk Game

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January 2011

Abstract

In the Crawford-Sobel (uniform, quadratic utility) cheap-talk model, we consider a simple mediation scheme (a communication device) in which the informed agent reports one of $N$ possible elements of a partition to the mediator and then the mediator suggests one of $N$ actions to the uninformed decision-maker according to the probability distribution of the device. We show that such a simple mediated equilibrium cannot improve upon the unmediated $N$-partition Crawford-Sobel equilibrium when the preference divergence parameter (bias) is small.

Keywords: Cheap Talk, Mediated Equilibrium.

JEL Classification Numbers: C72.

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*The question analysed in this paper stemmed out of a conversation with Marco Ottaviani. We wish to thank all seminar and conference participants at Bath, Belfast, Birmingham, Caen, CORE, ESWC 2005, GTWC 2004, ISI Kolkata, Keele and Nottingham for helpful comments and particularly, Jayasri Dutta, Herakles Polemarchakis, and Ashoke Sinha for constructive suggestions.

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1 INTRODUCTION

We consider the (uniform, quadratic utility) cheap-talk model of Crawford and Sobel (1982) (hereafter referred to as the CS model) to study the effect of mediation in comparison to unmediated cheap-talk communication. In this game, the strategic interaction and information-transmission between an uninformed decision maker (called the receiver) and an informed agent (called the sender) has been studied and what can be achieved by unmediated cheap-talk communication is well established. CS have proved that any (Bayesian-Nash) equilibrium in this cheap-talk game is equivalent to a partition equilibrium where the informed agent reveals one of the finitely many elements of the partition in which the true state of the nature lies. The number of elements in the most informative partition equilibrium in the CS model, $N$, depends on the value of the preference divergence parameter, $b$.

Krishna and Morgan (2004) introduced a new unmediated communication protocol which improved the welfare of the two players relative to the CS equilibrium. Krishna and Morgan also constructed an example of mediated communication in the CS model demonstrating the possibility of Pareto improvement.

Blume, Board and Kawamura (2007) consider the effect of adding noise to the sender’s message in the Crawford-Sobel model. The noise can be interpreted as communication error. Also, one can think of their communication scheme as a special kind of a mediator who passes on messages from the sender to the receiver with some exogenous noise added on. Specifically, with some probability, the mediator passes on the sender’s message to the receiver unchanged; otherwise, independent of the message sent by the sender, the mediator passes on a random message from some fixed error distribution. They show that for a sufficiently small amount of noise, it is possible to improve upon the $N$-partition CS equilibrium as long as $b \neq \frac{1}{2N}$ for any integer $N$ and $b < \frac{1}{2}$.

Goltsman, Horner, Pavlov and Squintani (2009) also analyze mediation in the cheap-talk framework of CS. They derive the optimal unconstrained mediated mechanism and an upper bound on the receiver’s payoff that can be
achieved by any mediated equilibrium. This upper bound is achieved by the construction of Blume et al (2007) when the level of noise is chosen appropriately. This upper bound is also the same as the value of the expected payoff to the receiver that is achieved by the equilibrium of the modified CS game constructed by Krishna and Morgan when $b < \frac{1}{8}$.

Here, we should point out that in all the above three models, welfare-improving mediation or noise involves larger number of messages compared to the minimal number of messages required in the most informative CS equilibrium. None of the papers mentioned above addresses this issue of whether welfare-improving (non-strategic) mediation must necessarily involve additional messages and more actions induced in equilibrium.

We, in this paper, consider mediation schemes\footnote{Ivanov (2009) considers the role of a strategic mediator in the CS framework. He shows that, for any bias $b$, there exists a strategic mediator who can help achieve the optimal payoffs obtained through a non-strategic mediator. Dessein (2002) considered delegation to an intermediary in the CS framework. However, the role of his “intermediary” is different from that of “mediation” in our context. Kovac and Mylovanov (2009) studied the relative performance of noisy or stochastic mechanisms and deterministic mechanisms in a very similar principal-agent setting.} in which the informed agent reports one possible element of a partition to a mediator (a communication device) and then the mediator suggests an action to the uninformed decision-maker according to the probability distribution of the device. We ask the question whether it is sufficient to use simple schemes (involving the same number of messages, $N$, as in the most informative CS equilibrium) for mediation to be Pareto superior to the CS equilibrium.

In particular, we concentrate on a specific form of mediated equilibria that we call $N$-simple mediated equilibria in which the mediator is restricted to use the same number ($N$) of inputs and outputs as the number of elements of the $N$-partition CS equilibrium. This clearly imposes a restriction on the mediated mechanisms that one may consider but this restriction is made in order to enable us to answer the above question. The mediator, associated with a specific probability distribution, can be interpreted as a communication scheme that the
players mutually agree to use. In this scheme, the possible number of elements the sender is allowed to use and the receiver is expected to receive via the mediator is restricted to \( N \), as in the most informative CS equilibrium. Note that \( b \), and hence \( N \), is commonly known to the players.

Our set-up of cheap-talk and mediation can be used to understand real world scenarios, for example, one involving a politician (an uninformed decision-maker), a civil servant (a mediator) and an expert scientist (an informed agent). We are not formally modelling such a situation, in particular, the role of a civil servant. However, one can certainly consider a situation in which the expert meets the civil servant and reports (stochastically) the true state of the nature and the civil servant in turn suggests an action (again, probabilistically) to the politician. In such a set-up, it is natural to presume that the civil servant would use a message space that is not richer or more complex than what would be used in direct communication between the expert and the decision-maker.

The main question we are asking in this paper can be translated as whether a “simple” civil servant is worth having in a conversation between a politician and an expert.

The result of our paper (Theorem 1) is that the \( N \)-partition CS equilibrium cannot be improved upon by the corresponding \( N \)-simple mediated equilibrium when the preference divergence parameter \( b \) is small (less than \( \frac{1}{2N} \)). In other words, when \( b \) is small, mediation needs to use more messages (relative to the minimal number of messages that can be used in the best CS equilibrium) in order to improve upon the \( N \)-partition CS equilibrium. If mediation or noise is simply a randomization over the messages that the sender would have reported in an original unmediated CS equilibrium, then it cannot improve information transmission, when the degree of preference misalignment is small.

This result advances our understanding of the effect of mediation in the CS model. It partially answers the question of whether the minimal size of the message space needs to be larger for mediation to Pareto dominate the CS equilibrium. Although we do not answer the question whether an \( N \)-simple mediated equilibrium can always improve on the CS equilibrium for large \( b \), we
provide a suggestive example.

Our theorem can be further interpreted in two ways. First, we establish that in some cases, it is not possible to improve upon the Bayesian Nash payoffs by (simple) mediation. This is in contrast with the notion of correlated equilibria\(^2\) (Aumann 1974, 1987) that typically improve upon Nash equilibria of any normal form game. Second, our main result identifies a class for which the unmediated equilibrium can not be improved upon, as in Moulin and Vial (1978). However, we would like to note that our theorem contrasts with the conclusion of Moulin and Vial (1978) who show that mediated equilibria can not improve upon Nash equilibria for a class of games that are strategically zero-sum, i.e., games where the conflict of interest between players is large. We, on the other hand, show that our restricted class of mediated mechanisms can not improve upon the Bayesian-Nash equilibria of the CS model when the conflict of interest is sufficiently small.

\section{THE MODEL}

\subsection{Crawford-Sobel Game}

Our set-up is identical to the uniform-quadratic utility CS Model, as presented in the literature. Informed readers may wish to skip this subsection.

There are 2 agents. The informed agent, called the sender \((S)\), precisely knows the state of the world, \(\theta\), where \(\theta \sim U[0,1]\), and can send a message at no cost, based on his private information, to the other agent, called the receiver \((R)\). The receiver, however doesn’t know \(\theta\) but must choose some decision \(y\) based on the information contained in the signal. The receiver’s payoff is \(U^R(y, \theta) = -(y - \theta)^2\), and the sender’s payoff is \(U^S(y, \theta, b) = -(y - (\theta + b))^2\), where \(b > 0\) is a parameter that measures the ‘bias’ in their preferences.

CS have shown that any equilibrium of this game is essentially equivalent to a partition equilibrium where only a finite number of actions are chosen in

\(^2\)The notion of correction can be suitably extended to finite games with incomplete information (Forges 1993, 2006).
equilibrium and each action corresponds to an element of the partition. For \( b < \frac{1}{2N(N+1)} \), where \( N \geq 2 \) is an integer, there is an equilibrium\(^3\) in which the state space is partitioned into \( N \) elements, characterised by 
\[
0 = a_0 < a_1 < a_2 < \ldots < a_{N-1} < a_N = 1,
\]
where \( a_k = \frac{k}{N} + 2bk(k - N) \), in which \( S \) sends a message for each element \([a_{k-1}, a_k)\), and given this message, \( R \) takes the optimal action \( y_k = \frac{a_{k-1} + a_k}{2} \). We call this the \( N \)-partition CS equilibrium. For \( \frac{1}{2N(N+1)} \leq b < \frac{1}{2(N(N-1))} \), the “best” equilibrium (the one that maximises \( EU^R \)) is the \( N \)-partition CS equilibrium\(^4\). For such an equilibrium, the receiver’s expected payoff is 
\[
EU^R = -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{8},
\]
while the sender’s expected payoff is 
\[
EU^S = EU^R - b^2.
\]

2.2 Mediated Equilibrium

Within the CS framework, we now consider mediation, a possible structure of which could be as follows: \( S \) sends a message based on his private information to the mediator; the mediator then chooses an action according to a commonly-known probability distribution and recommends it to \( R \). We here consider a specific form of mediation (mechanism) as formally defined below.

**Definition 1** An \( N \times M \) mediated talk is \((\{x_k\}_{k=0}^N, \{y_j\}_{j=1}^M, \{p_{kj}\}_{k=1,\ldots, N; j=1,\ldots, M})\) where 
\[
0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = 1,
\]
each \( y_j \in [0,1] \) for \( j \in \{1,2,\ldots,M\} \), each \( p_{kj} \in [0,1] \) for \( k \in \{1,2,\ldots,N\} \), \( j \in \{1,2,\ldots,M\} \) with 
\[
\sum_j p_{kj} = 1.
\]

In an \( N \times M \) mediated talk, \( S \) reports one of \( N \) possible elements, \([x_{k-1}, x_k)\), in which the true state \( \theta \) may lie, to the mediator, and given the report \( \theta \in [x_{k-1}, x_k) \), the mediator then recommends to \( R \) one action, \( y_j \), out of \( M \) possible actions, with probability \( p_{kj} \).

\(^3\)For \( \frac{1}{4} \leq b \), babbling is the only equilibrium.  
\(^4\)Note that for any \( b < \frac{1}{2(N(N-1))} \), an \( N \)-partition CS equilibrium does exist. However, it is not the “best”. Chen, Kartik and Sobel (2008) provide a formal selection argument for the “best” equilibrium.
Such a mechanism\textsuperscript{5} is said to be in equilibrium if it is incentive compatible for both players, that is, if (i) $S$ has the incentive to be truthful to the mediator given the probabilities $p_{kj}$, and (ii) $R$ has the incentive to obey the mediator's recommendation $y_j$, given the posterior probabilities on the state of nature.

In what follows, we will focus only on $N \times N$ mediated equilibria. We will call such an equilibrium an $N$-simple mediated equilibrium. Formally,

**Definition 2** For any specific value of $b$, an $N \times N$ (or, $N$-simple) mediated equilibrium is an $N \times M$ (with $M = N$) mediated talk that satisfies

(i) incentive compatibility for $S$: $f_k'(\theta) > 0$ and $f_k(x_k) = 0$ for all $k \in \{1, 2, ..., N - 1\}$, where, $f_k(\theta) = \sum_{j=1}^{N} (p_{kj} - p_{k+1j}) [y_j - (\theta + b)]^2$, and

(ii) incentive compatibility for $R$: $y_j = \arg\max_y \sum_{k=1}^{N} q_{kj} \frac{1}{(x_k - x_{k-1})} \int_{x_{k-1}}^{x_k} (y - \theta)^2 d\theta$ for all $j \in \{1, 2, ..., N\}$, where, $q_{kj}$ is the posterior probability that $\theta \in [x_{k-1}, x_k)$ and is given by $q_{kj} = \frac{(x_k - x_{k-1})p_{kj}}{\sum_{k=1}^{N} (x_k - x_{k-1})p_{kj}}$.

In (i), $f_k(\theta)$ is the difference in expected utility from inducing the distributions $\{p_{k+1j}\}_{j=1,..,N}$ and $\{p_{kj}\}_{j=1,..,N}$ that a type $\theta$ sender would obtain.

$f_k(x_k) = 0$ essentially corresponds to the “arbitrage condition” in the CS equilibrium. Incentive compatibility for the sender requires that the $x_k$ type sender is indifferent between inducing $\{p_{k+1j}\}_{j=1,..,N}$ and $\{p_{kj}\}_{j=1,..,N}$. Also a sender with type $\theta \in (x_k, x_{k+1})$ should prefer the distribution $\{p_{k+1j}\}_{j=1,..,N}$ over $\{p_{kj}\}_{j=1,..,N}$ implying that $f_k'(\theta) > 0$.

In (ii), incentive compatibility for $R$ requires that when $y_j$ has been recommended, $R$ indeed chooses the action $y_j$ because it maximizes his expected utility given his posterior beliefs.

### 2.3 Characterisation

An $N$-simple mediated equilibrium can be characterised easily. Incentive compatibility for $R$, as in Definition 2(ii), requires

\textsuperscript{5}This is the type of mechanism Krishna and Morgan (2004) also considered to construct an example in their paper. Myerson (website) used a “discrete” version of such a mechanism.
\[
\sum_{k=1}^{N} p_{kj} \left[ (y_j - x_{k-1})^2 - (y_j - x_k)^2 \right] = 0; \text{ for all } j = 1, \ldots, N,
\]

which implies

\[
y_j = \frac{1}{2} \left[ \frac{\left( 1 - \sum_{k \neq j} p_{jk} \right) (x_j^2 - x_{j-1}^2) + \sum_{k \neq j} p_{kj} (x_k^2 - x_{k-1}^2)}{\left( 1 - \sum_{k \neq j} p_{jk} \right) (x_j - x_{j-1}) + \sum_{k \neq j} p_{kj} (x_k - x_{k-1})} \right].
\]

Incentive compatibility for \(S\), as in Definition 2(i), requires

\[
\sum_{j=1}^{N} (p_{kj} - p_{k+1j}) \left[ (y_j - (x_k + b))^2 = 0; \text{ for all } k = 1, \ldots, N - 1,
\]

and

\[
\sum_{j=1}^{N-1} (p_{k+1j} - p_{kj}) (y_j - y_N) > 0; \text{ for all } k = 1, \ldots, N - 1.
\]

Using (3), we get

\[
2 (x_k + b) = \frac{\sum_{j=1}^{N-1} (p_{kj} - p_{k+1j}) (y_j^2 - y_N^2)}{\sum_{j=1}^{N-1} (p_{kj} - p_{k+1j}) (y_j - y_N)}
\]

\[
= \frac{\left( 1 - \sum_{j \neq k} p_{kj} - p_{k+1k} \right) \left( y_k^2 - y_N^2 \right) + \left( p_{kk+1} - 1 + \sum_{j \neq k+1} p_{k+1j} \right) (y_{k+1}^2 - y_N^2)}{\left( 1 - \sum_{j \neq k} p_{kj} - p_{k+1k} \right) (y_k - y_N) + \left( p_{kk+1} - 1 + \sum_{j \neq k+1} p_{k+1j} \right) (y_{k+1} - y_N)}
\]

for all \(k = 1, \ldots, N - 1\).

Thus an \(N\)-simple mediated equilibrium is characterised by (5) where the \(y_j\)'s are given by (2) with the constraints that the inequalities in (4) are satisfied.
3 THE RESULT

To state and prove our result, we take $b < \frac{1}{2N(N-1)}$, for which the $N$-partition CS equilibrium exists.

We first observe that the $N$-partition CS equilibrium is equivalent to an $N$-simple mediated equilibrium. The $N$-partition CS equilibrium as described in Section 2.1 can be equivalently written as an $N \times N$ mediated talk with $x_k = a_k = \frac{k}{N} + 2bk(k-N)$ for all $k \in \{1,...N\}$; $y_j = \frac{a_j-1+a_j}{2}$ for all $j \in \{1,...N\}$; $p_{kj} = 0$ for all $k, j \in \{1,...N\}$, $k \neq j$. This is indeed an $N$-simple mediated equilibrium as the incentive compatibility condition for $R$, given by (2), and the incentive compatibility condition for $S$, given by (5) are satisfied.

We now state and prove our main result which answers the question when the $N$-partition CS equilibrium cannot be improved upon by an $N$-simple mediated equilibrium. Note that, in the class of mediated equilibria, $EU^S = EU^R - b^2$. Thus, ex-ante, the sender’s and receiver’s welfare ranking agree.

**Theorem 1** For $b < \frac{1}{2N^2}$, an $N$-simple mediated equilibrium cannot improve upon the $N$-partition CS equilibrium.

**Proof.** We need to prove that for $b < \frac{1}{2N^2}$, the maximum expected utility of $R$ from any $N$-simple mediated equilibrium is the same as in the $N$-partition CS equilibrium. Note again that the $N$-partition CS equilibrium corresponds to the $N$-simple mediated equilibrium with $p_{kj} = 0$ for all $k \neq j; k, j \in \{1,...N\}$.

Hence, it suffices to show that, for $b < \frac{1}{2N^2}$, the $N$-simple mediated equilibrium which maximizes the expected utility of $R$ corresponds to $p_{kj} = 0$ for all $k \neq j; k, j \in \{1,...N\}$.

Now note that

$$EU^R = -\sum_{k=1}^{N} \left[ \left( 1 - \sum_{j \neq k} p_{kj} \right) \int_{x_{k-1}}^{x_k} (y_k - \theta)^2 d\theta + \sum_{j=1}^{N} p_{kj} \int_{x_{k-1}}^{x_k} (y_j - \theta)^2 d\theta \right]$$

(6)

$$= -\frac{1}{3} \sum_{k=1}^{N} \left[ \left( 1 - \sum_{j \neq k} p_{kj} \right) \left[ (y_k - x_{k-1})^3 - (y_k - x_k)^3 \right] + \sum_{j=1}^{N} p_{kj} \left[ (y_j - x_{k-1})^3 - (y_j - x_k)^3 \right] \right]$$

(7)
To prove our result, we now consider the following constrained maximization problem.

Maximize \( \{x_k\}_{1 \leq k \leq N}, \{p_{kj}\}_{k \neq j} \) subject to (5) where the \( y_j \)'s in (5) are given by (2)

We then prove that, for \( b < \frac{1}{2N^2} \), the solution of the above problem is achieved at a “corner”, namely, \( p_{kj} = 0 \) for all \( k \neq j \); \( k, j \in \{1, ..., N\} \) and \( x_k = x_k^{CS} = \frac{b}{\beta} + 2b(k - N) \) for all \( k \in \{1, ..., N - 1\} \).

This part of the proof has been relegated to the Appendix.

To complete the proof, note that we here have dropped the constraint that (4) be satisfied and found the optimal solution of this modified constrained maximization problem with a larger “feasible set”. However, notice that the above optimal solution, namely, the CS \( N \)-partition equilibrium, does satisfy the constraint (4) and hence will be the solution of the desired maximization problem.

Hence the theorem. ■

4 REMARKS

Our result can be illustrated easily using examples with \( N = 2 \). Recall that a 2-partition CS equilibrium exists for \( b < \frac{1}{4} \) and for \( \frac{1}{12} \leq b < \frac{1}{4} \), the best CS partition equilibrium involves two elements.

A 2-simple mediated equilibrium is given by\( ^6 \) \((x, y_1, y_2, p_{11}, p_{12}, p_{21}, p_{22})\), where, \( x, y_1, y_2 \in (0, 1), p_{11}, p_{12}, p_{21}, p_{22} \in [0, 1] \) and \( p_{11} + p_{12} = 1, p_{21} + p_{22} = 1 \).

The incentive compatibility constraints for \( S \) and for \( R \) can all be combined into the following equation:

\[
\frac{(1 - p_{12})x^2 + p_{21}(1 - x^2)}{4[(1 - p_{12})x + p_{21}(1 - x)]} + \frac{p_{12}x^2 + (1 - p_{21})(1 - x^2)}{4[p_{12}x + (1 - p_{21})(1 - x)]} - x = b \quad (8)
\]

\(^6\)We drop the subscript in \( x_1 \) for presentational simplicity.
Thus, a 2-simple mediated equilibrium in this set-up can be characterised by three variables \((p_{12}, p_{21}, x)\), where, \(x \in (0, 1)\) and \(p_{12}, p_{21} \in [0, 1]\) satisfying (8). We can now use the above to construct examples.

Consider, for example, \(b = \frac{1}{10}\). Here, the 2-partition CS equilibrium is characterised by \(y_1 = \frac{3}{20}, y_2 = \frac{13}{20}\) and \(a = \frac{3}{10}\) with utilities \(EU^R = -\frac{37}{1200} \approx -0.031\) and \(EU^S = -\frac{49}{1200} \approx -0.041\). It is easy to check that \(y_1 = \frac{19}{100}, y_2 = \frac{31}{100}, x = \frac{4}{5}\) with \(p_{11} = \frac{4}{5}, p_{12} = \frac{1}{5}, p_{21} = \frac{4}{15},\) and \(p_{22} = \frac{7}{15}\) constitute a 2-simple mediated equilibrium with utilities \(EU^R = -\frac{517}{7500} \approx -0.069\) and \(EU^S = -\frac{592}{7500} \approx -0.079\); however, it does not improve upon the 2-partition CS equilibrium. Indeed, our result confirms that the 2-partition CS equilibrium for \(b = \frac{1}{10}\) (for any \(b < \frac{1}{4}\)) cannot be improved upon by any 2-simple mediated equilibrium.

Our result suggests that an \(N\)-simple mediated equilibrium may improve upon the \(N\)-partition CS equilibrium when \(b\) is large enough \((\frac{1}{2N} \leq b < \frac{1}{2N(N-1)}\); for \(N = 2, \frac{1}{8} \leq b < \frac{1}{4}\)). We illustrate this for \(b = \frac{1}{6}\). Here, the 2-partition CS equilibrium is characterised by \(a = \frac{1}{6}, y_1 = \frac{1}{12},\) and \(y_2 = \frac{7}{12}\) with utilities \(EU^R = -\frac{7}{144} \approx -0.0486\) and \(EU^S = -\frac{11}{144} \approx -0.0764\). It is easy to check that \(x = 0.2245201023, y_1 = 0.1745967377, y_2 = 0.6077768002, p_{11} = 0.97,\) and \(p_{21} = 0.04\) constitute a 2-simple mediated equilibrium with utilities \(EU^R \approx -0.0483\) and \(EU^S \approx -0.0760\) and can improve upon the corresponding 2-partition CS equilibrium.

It is also worth pointing out that for a large \(N\) (corresponding to a small \(b\)), it is harder to construct an \(N\)-simple mediated equilibrium (as above) that improves upon the corresponding \(N\)-partition CS equilibrium, as the necessary range of \(b\) \((\frac{1}{2N} \leq b < \frac{1}{2N(N-1)})\) becomes smaller.

For a fixed \(N\) however, the \(N\)-partition CS equilibrium can be improved upon, even when \(b\) is small, by a mediated equilibrium that is more complex, involving higher number of messages and actions, than our \(N\)-simple mediated equilibrium.

Formally, one can prove that the \(N\)-partition CS equilibrium can be improved upon by an \((N + 1) \times (N + 2)\) mediated equilibrium when \(b < \frac{1}{8}\) by
using the construction of Krishna and Morgan (2004). The equilibrium constructed by Krishna and Morgan (2004) (call it the \( \mathcal{E} \)-KM equilibrium) exists when \( \frac{1}{2(N+1)} < b < \frac{1}{2N} \) and \( b < \frac{1}{8} \) and improves upon the corresponding \( \mathcal{E} \)-partition CS equilibrium. It is also easy to prove that the \( \mathcal{E} \)-KM equilibrium, when it exists (i.e., for \( b < \frac{1}{8} \)), is equivalent to an \( (\mathcal{E}+1) \times (\mathcal{E}+2) \) mediated equilibrium. We further know, due to Goltsman et al (2009), that the payoffs corresponding to the \( \mathcal{E} \)-KM equilibrium are the same as those in the optimal mediated equilibrium. Therefore, one can conclude that an \( (\mathcal{E}+1) \times (\mathcal{E}+2) \) mediated equilibrium achieves the optimal payoffs from mediation in the CS model and cannot be improved upon by any other mediated mechanism whenever \( b < \frac{1}{8} \).

For example, with \( b = \frac{1}{40} \), the \( 3 \times 4 \) mediated equilibrium characterised by \( x_1 = \frac{2}{3}, x_2 = \frac{4}{3}, y_1 = \frac{1}{3}, y_2 = \frac{2}{3}, y_3 = \frac{5}{3}, y_4 = \frac{7}{3}, p_{11} = 1, p_{22} = \frac{5}{9}, p_{24} = \frac{4}{9}, p_{33} = \frac{5}{9}, \) and \( p_{34} = \frac{4}{9} \) with \( EU^R = -\frac{36}{120} = -0.3 \) and \( EU^S = -\frac{48}{120} = -0.4 \) improves upon the corresponding 2-partition CS equilibrium and also is the best mediated equilibrium in this case.

5 APPENDIX

Proof of Theorem 1 (contd.): Consider the Lagrangian:

\[
\mathcal{L} = -\sum_{k=1}^{N} \left[ (1 - \sum_{j \neq k} p_{kj}) \left( (y_k - x_{k-1})^3 - (y_k - x_k)^3 \right) + \sum_{j=1}^{N} p_{kj} \left( (y_j - x_{j-1})^3 - (y_j - x_j)^3 \right) \right]
\]

\[
+ \sum_{k=1}^{N-1} \lambda_k \left[ 2b - \frac{1}{(1 - \sum_{j \neq k} p_{kj} - p_{k+1k})(y_k^2 - y_N^2)} \right]
\]

To complete the proof, we just need to show that at the proposed solution,
there exist \( \{\lambda_k\}_{k=1}^{N-1} \) such that \( \frac{\partial C}{\partial x_k} = 0 \) for all \( k = 1, \ldots, N-1 \) and \( \frac{\partial C}{\partial p_{kj}} < 0 \) for all \( j \neq k \), when \( b < \frac{1}{2N} \).

First, at \( p_{kj} = 0 \) for all \( k \neq j; \ k, j \in \{1, \ldots, N\} \), it is easy to check that \( \frac{\partial y_k}{\partial x_k} = \frac{\partial y_k}{\partial x_{k-1}} = \frac{1}{2} \), and \( \frac{\partial y_k}{\partial x_j} = 0 \) for all \( j \neq k, k - 1 \). Also, \( \frac{\partial y_k}{\partial p_{ki}} = \frac{(x_k-x_{k-1})(x_k+x_{k-1}-x_j-x_{j-1})}{2(x_j-x_{j-1})} \) for all \( k \neq j \) and \( \frac{\partial y_k}{\partial p_{ki}} = 0 \) for all \( l \neq j \), for all \( k \).

Subsequently, it can be shown that \( \frac{\partial C}{\partial x_k} = -3 \left[(y_k-x_k)^2-(y_{k+1}-x_k)^2\right] + \lambda_k - \frac{\lambda_{k+1}}{2} - \frac{\lambda_{k-1}}{2} \) for all \( k = 1, \ldots, N-1 \) (since \( \lambda_0 \) and \( \lambda_N \) are not defined, define them to be equal to zero).

Now \( \frac{\partial C}{\partial x_k} = 0 \) implies \( 12b(y_{k+1} - y_k) + 2\lambda_k - \lambda_{k+1} - \lambda_{k-1} = 0 \) for all \( k = 1, \ldots, N-1 \).

This gives us a system of \( N-1 \) equations in \( N-1 \) variables, \( \lambda_1, \ldots, \lambda_{N-1} \), which can be succinctly written in matrix form as

\[
\begin{bmatrix}
2 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\vdots \\
\lambda_{N-1}
\end{bmatrix}
= 
\begin{bmatrix}
12b(y_1 - y_2) \\
12b(y_2 - y_3) \\
12b(y_3 - y_4) \\
\vdots \\
12b(y_{N-1} - y_N)
\end{bmatrix}
\]

The \((N-1) \times (N-1)\) matrix above is symmetric and tridiagonal, the \(ij\)-th element of the inverse of which is given by \( \frac{1}{2N}(i+j-|j-i|)(2N-|j-i|-i-j) \) (using results by Hu and O’Connell 1996 and Yamani and Abdelmonem, 1997).

Thus, solving the equations, we get, \( \lambda_k = -\frac{2bk(N-k)}{N}[3 - 2bN^2 + 4bkN] \) (which is < 0 for all \( k = 1, \ldots, N-1 \)).

We are now ready to show that, when \( b < \frac{1}{2N} \), \( \frac{\partial C}{\partial p_{kj}} < 0 \) for all \( j \neq k \), at the proposed solution (the CS equilibrium values of \( x_k \)'s and \( y_k \)'s) and with the above values \( \lambda_k \) for all \( k = 1, \ldots, N-1 \).

For all \( j \neq k \), we have
\[
\frac{\partial L}{\partial p_{kj}} = \left[ (y_k - x_{k-1})^3 - (y_k - x_k)^3 \right] - \left[ (y_j - x_{k-1})^3 - (y_j - x_k)^3 \right] \\
- \lambda_k \left[ \frac{(y_j - y_k)(y_j - y_{k+1})}{(y_k - y_{k+1})} \right] - \lambda_{k-1} \left[ \frac{(y_k - y_j)(y_j - y_{k-1})}{(y_{k-1} - y_k)} \right] \\
- (\lambda_j + \lambda_{j-1}) \left[ \frac{\partial y_j}{\partial p_{kj}} \right]
\]

We first prove \( \frac{\partial L}{\partial p_{kj}} < 0 \) for \( k = 1 \), when \( b < \frac{1}{2N^2} \). Here,

\[
\frac{\partial L}{\partial p_{1j}} = \frac{(1 - j)^2(2bN^2 + 1)(2bN^2 - 2bjN - 1)(12b^2N^4 - 12b^2N^3)}{N^3(2bN^2 + 2bN - 4bjN - 1)(2bN^2 - 4bN - 1)}
\]

Clearly, (i) \((1 - j)^2 > 0\); (ii) \((2bN^2 + 1) > 0\); and (iii) \(N^3 > 0\). Also note that as \( b < \frac{1}{2N^2(N-1)} \), we have \(2bN^2 - 2bN - 1 < 0\). It is now easy to check that (iv) \(2bN^2 - 2bjN - 1 < 0\); (v) \(2bN^2 + 2bN - 4bjN - 1 < 0\); and (vi) \(2bN^2 - 4bN - 1 < 0\).

Finally, the factor,

\[
(1 - j)^2(2bN^2 + 1)(2bN^2 - 2bjN - 1)(12b^2N^4 - 12b^2N^3)
\]

which we need to show is \(> 0\) for all \( j \geq 2 \). Clearly, it is so for all \( j \geq 3 \). For \( j = 2 \), the factor is equal to \(3(2N^2b - 1)^2 + 6Nb(1 + j)(2Nb + 1 - 2N^2b) + 4N^2b^2[(j - 2)^2 - 7 + j^2]\) which can be shown to be \(> 0\) whenever \( b < \frac{1}{2N^2 - 4b} \). Since \( \frac{1}{2N^2 - 4b} > \frac{1}{2N^2} \), we have that the factor is for \(> 0\) all \( j \geq 2 \) when \( b < \frac{1}{2N^2} \). Hence, \( \frac{\partial C}{\partial p_{ij}} < 0 \) for all \( j \geq 2 \), when \( b < \frac{1}{2N^2} \).

Now we show that \( \frac{\partial C}{\partial p_{kj}} < 0 \) for all \( k > 1 \), when \( b < \frac{1}{2N^2} \). Substituting the values for the \( x_k \)'s, \( y_k \)'s and \( \lambda \)'s, we have,
\[
\frac{\partial L}{\partial p_{kj}} = \frac{-12bN^2 + 18bkN + 6bjN - 12bN + 3}{N^3(2bN^2 + 2bN - 4bjN - 1)(2bN^2 + 4bN - 4bkN - 1)}
\]

Here clearly, (i) \((k - j)^2 > 0\); (ii) \((2bN^2 + 1) > 0\); and (iii) \(N^3 > 0\). Once again, as \(b < \frac{1}{2N(N - 1)}\), we have \(2bN^2 - 2bN - 1 < 0\). Thus one can verify that

(iv) \(2bN^2 + 2bN - 2bjN - 2bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(2 - j - k) < 0\); (v) \(2bN^2 + 2bN - 4bjN - 1 = 2bN^2 - 2bN - 1 + 4bN(1 - j) < 0\); (vi) \(2bN^2 + 4bN - 4bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(3 - 2k) < 0\) as \(k \geq 2\); and (vii) \(2bN^2 - 4bkN - 1 = 2bN^2 - 2bN - 1 + 2bN(1 - 2k) < 0\).

Finally, note that the factor,

\[
12b^2N^4 - 36b^2kN^3 - 12jb^2N^3 + 24b^2N^3 + 32b^2k^2N^2 + 8jb^2N^2k + 4b^2N^2k - 12b^2jN^2 + 8b^2j^2N^2
-12bN^2 + 18bkN + 6bjN - 12bN + 3
= 12N^4b^2 - 12N^3b(3k + j - 2) + 4N^2b^2[8k^2 + 2jk + 1 - 9k - 3j + 2j^2]
-12N^2b + 6Nb(3k + j - 2) + 3
= [12N^4b^2 - 12N^2b + 3]
+(3k + j - 2)[6Nb - 12N^3b^2] + 12N^2b^2(3k + j - 2)
+4N^2b^2[8k^2 + 2jk + 1 - 9k - 3j + 2j^2] - 12N^2b^2(3k + j - 2)
= 3(2bN^2 - 1)^2 + 6bN(3k + j - 2)[1 + 2bN - 2bN^2]
+4N^2b^2[(k + j - 3)^2 + 7k(k - 2) + 2(k - 1) + j^2]
\]

which is \(> 0\) as \(1 + 2bN - 2bN^2 > 0\) (as \(b < \frac{1}{2N(N - 1)}\)) and \(k \geq 2\).

Hence, \(\frac{\partial L}{\partial p_{kj}} < 0\) for all \(j \neq k\), and for all \(k > 1\), when, \((2bN^2 - 1) < 0\), i.e., when, \(b < \frac{1}{2N^2}\).
6 REFERENCES


