Imperfect Knowledge about Asset Prices and Credit Cycles
Imperfect Knowledge about Asset Prices and Credit Cycles

Pei Kuang *
University of Birmingham

November 20, 2013

Abstract

This paper develops an equilibrium model with a housing collateral constraint in which rational agents are uncertain about the collateral price process. Bayesian learning by agents can endogenously generate booms and busts in collateral prices and significantly strengthen the role of the collateral constraint as an amplification mechanism through the interaction of agents’ price beliefs, price realizations and credit limits. Over-optimism or pessimism is fueled when a surprise in price expectations is interpreted partially by the agents as a permanent change in the parameters governing the collateral price process and is validated by subsequently realized prices. The learning model can quantitatively account for the recent US boom-bust cycle in house prices, household debt and aggregate consumption dynamics during 2001-2008. The paper also demonstrates that the leveraged economy with a higher steady state leverage ratio is more prone to self-reinforcing learning dynamics.

Keywords: Boom-Bust, Collateral Constraints, Learning, Leverage, Housing
JEL classifications: D83, D84, E32, E44

*Corresponding author at: JG Smith Building, Department of Economics, University of Birmingham, UK B152TT. E-mail address: P.Kuang@bham.ac.uk. Tel: (+44) 121 4145620.
“At some point, both lenders and borrowers became convinced that house prices would only go up. Borrowers chose, and were extended, mortgages that they could not be expected to service in the longer term. They were provided these loans on the expectation that accumulating home equity would soon allow refinancing into more sustainable mortgages. For a time, rising house prices became a self-fulfilling prophecy, but ultimately, further appreciation could not be sustained and house prices collapsed.” (Bernanke (2010))

1 Introduction

The recent decade has witnessed a massive run-up and subsequent collapse of house prices, as well as the remarkable role of the interaction of housing markets and credit markets in aggregate fluctuations in the U.S. economy. Real house prices increased considerably in the decade before the recent financial crisis, as seen in the upper panel of figure 1.\(^1\) They displayed relatively smaller variability before the year 2000 and increased by 35.9\% from 2001 to 2006 in which house prices peaked. Associated with the price boom was a sharp increase in the household credit market debt/GDP ratio\(^2\) and a consumption boom. As can be seen from the lower panel of figure 1, the household credit market debt/GDP ratio changed moderately before the year 2000 but increased from 45\% in 2001 to 70\% in 2006. Aggregate consumption\(^3\) grew at about 3\% per annum between 2001 and 2006, while its growth dropped sharply after house prices started to revert, as shown in figure 2.

Building upon the model of Kiyotaki and Moore (1997, henceforth KM), the paper develops a dynamic general equilibrium model with a housing collateral constraint and learning that can quantitatively account for the recent U.S. boom-bust in house prices, and associated household credit market debt and aggregate consumption dynamics during 2001-2008 following the strong fall in real interest rates after the year 2000.

Much of recent research attempting to understand the recent house price dynamics include a housing collateral constraint, such as Boz and Mendoza (2013), Ferrero (2011), Justiniano, Primiceri, and Tambalotti (2012), and Hoffmann, Krause and Laubach (2012). Despite the critical role in the recent financial and macroeconomic turmoil, the massive run-up of house prices is extremely difficult to generate in most existing optimizing-agent DSGE models with housing collateral constraints. These models typically assume that economic agents could rationally foresee future collateral

\(^{1}\)The data is taken from the OECD. Its definition is the “national wide single family house price index”. The real house price index is the nominal house price index deflated by CPI price index. It is normalized to a value of 100 in 2000. The price-to-rent ratio and price-to-income ratio display a similar pattern.

\(^{2}\)The household credit market debt/GDP ratio is measured by the absolute value of the ratio of net credit market assets of US household and non-profit organizations to GDP. The data is from the Flow of Funds Accounts of the U.S. provided by the Board of Governors of the Federal Reserve System.

\(^{3}\)The data is from Federal Reserve System. It is the Real Personal Consumption Expenditures (series ID: PCECC96).
prices associated with any possible contingency. Therefore, the link between collateral prices and fundamentals is tight, while the latter has relatively smaller variability.

In contrast to the previous literature with housing collateral constraints, I assume that agents have an incomplete model of the macroeconomy, knowing their own objective, constraints and beliefs but not the equilibrium mapping between fundamentals (e.g. preference shocks, collateral holdings) of the economy and collateral prices. Instead, agents have a completely specified subjective belief system about the collateral price process and make optimal decisions. By extrapolating from historical patterns in observed data they approximate this mapping to forecast future collateral prices.

The Bernanke quotation at the beginning of the paper can be viewed as a rough statement of the mechanism of the learning model. More details of the mechanism are as follows. Consider a positive surprise in collateral price forecasts, say due to an unexpected interest rate reduction, which induces agents’ belief revision and more optimistic expectations about future collateral prices relative to the Rational Expectations (RE). Credit limits are relaxed and larger loans are granted by lenders based on the optimism. The enhanced borrowing capacity increases borrowers’ collateral demand and boosts collateral prices. Higher than expected price realizations partially validate agents’ optimism, which leads to further optimism and persistent increases in actual prices. Meanwhile, production and consumption expand due to shifts of collateral to more productive households.

At some point, the debt repayment becomes excessive and its negative effect dominates so that collateral prices will fall short of agents’ expectations. This sets a self-reinforcing decline in motion. Agents start to revise their beliefs downward and become
pessimistic about the future liquidation value of collateral. Credit limits are tightened due to the pessimism and shifts of collateral back to the lenders. The realized prices reinforce agents’ pessimism, inducing periods of persistent and downward adjustments of beliefs and actual prices. Eventually prices and quantities converge to the steady state.

The trigger of the price boom in the model is the persistently low interest rates after the year 2000. Responses of prices and quantities of the learning model are significantly amplified due to the dynamic interaction of agents’ beliefs, credit limits and price realizations. A widening household credit market debt/GDP ratio is generated due to both the house price boom and rising amounts of collateral holdings by households. Production and consumption amplification arise from shifts of collateral to more productive borrowers.

Intrinsic to a collateral-constrained economy is that changes of agents’ beliefs or borrowing capacities can have a critical effect on collateral prices. For example, increasing optimism enhances borrowing capacity, which boosts collateral demand and collateral prices. Higher than expected price realizations could then feedback to further optimism. I find that such self-reinforcing learning dynamics can happen in the leveraged economy with a sufficiently high steady state leverage ratio. This gives rise to possible larger fluctuations of prices and quantities in a collateral-constrained economy in contrast with some stock pricing models with learning in an endowment economy, e.g., Timmermann (1996), and in a production economy without collateral constraints, e.g., Carceles-Poveda and Giannitsarou (2008) which find a limited role of adaptive learning in asset pricing. Section 4.3.2 provides a further discussion.

The transmission mechanism is consistent with the findings of Iacoviello and Neri (2010), which estimate a DSGE model with a housing collateral constraint via Bayesian methods using data from 1965 to 2006. They find an important role of monetary factors
in housing cycles over the whole sample and an increasing role during the recent housing cycle. In addition, they find a nonnegligible spillover effect from housing markets to consumption over the whole sample and increasing importance of the effect in the recent housing cycle.

The learning model is also broadly consistent with observed survey expectations data during the recent housing cycle. Gelain, Lansing, and Mendicino (2013) derived house price forecasts from the future markets for the Case-Shiller house price index (where only the data from 2006 onwards are available) and showed that “the future market tends to overpredict future house prices when prices are falling.” The learning model generates exactly such pattern. The learning model is also compatible with the finding of Piazzesi and Schneider (2009) that during the recent U.S. housing cycle the “optimism” in the housing markets, i.e., the share of agents believing prices to increase further, co-moves positively with the house price level using the data from Michigan Survey of Consumers.

1.1 Related Literature

A strand of literature has studied the role of collateral constraints as an amplification mechanism transforming relative small shocks to the economy into large output fluctuations, such as KM, Kocherlakota (2000), Krishnamurthy (2003), Cordoba and Ripoll (2004) among others. Allowing agents to be uncertain about the link between prices and fundamentals, the learning model can generate additional non-fundamental fluctuations in collateral prices, which strengthens the role of collateral constraints as an amplification mechanism.

The paper relates to the literature which explores the role of shifting expectations in business cycle fluctuations. For example, Eusepi and Preston (2011) present a business cycle model with learning which improves the internal propagation of business cycle shocks and is consistent with many features of observed survey expectations data. Huang, Liu and Zha (2009) study implications of adaptive expectations in a standard growth model and find them promising in generating plausible labor market dynamics. Milani (2011) estimates a New Keynesian Model with adaptive learning incorporating survey data on expectations and finds a crucial role of expectational shocks as a major driving force of the U.S. business cycle.

The paper also relates to papers which study the role of self-referential learning in asset pricing or asset price booms and busts. Timmermann (1996) examines the role of learning about stock prices in an endowment economy. Carceles-Poveda and Giannitsarou (2008) study an asset pricing model with learning in a production economy with capital accumulation. Adam, Marcet, and Nicolini (2012) and Adam and Marcet (2010) develop learning models which can quantitatively replicate several major stock pricing facts, generating booms and busts in stock prices and matching agents’ return expectations as in survey data. Lansing (2010) examines a near-rational solution to Lucas-type asset pricing models and learning to generate intermittent stock bubbles and to match many quantitative features observed in the long-run U.S. stock market.
data. Branch and Evans (2011) examine stock market booms and crashes in an asset pricing model with learning about the conditional variance of a stock’s return.

Gelain, Lansing and Mendicino (2013) study a DSGE model with a housing collateral constraint and a subset of agents using moving-average rules to forecast future house prices. They find that this “significantly amplifies the volatility and persistence of house prices and household debt” and examines various policy options to dampen the excess volatilities. The paper is related to Adam, Kuang and Marcet (2011, henceforth AKM) which use an open economy asset pricing model with learning to account for the G7 house price and current account dynamics. A more detailed discussion of the relation to AKM is in section 4.3.1.

The rest of the paper is structured as follows. The next section presents the model and agents’ optimality conditions. In section 3, I discuss the equilibrium with imperfect knowledge, belief specification and optimal learning behavior of agents. The mechanism of the learning model is inspected in section 4. Quantitative results and further intuition of the learning model are presented in section 5. Section 6 concludes.

2 The Model

The model builds on the basic version of the KM model with the major difference of belief specification and expectation formation.

2.1 The Model Setup

There are two types of goods in the economy, durable assets, i.e., houses, and non-durable consumption goods, which are produced using houses but cannot be stored. The durable assets play a dual role: they are not only factors of production but also serve as collateral for getting loans. There are two types of infinitely lived risk-neutral agents, households and financial intermediaries, each of which has unit mass. Both produce and eat consumption goods. At each date t, there are two markets. One is a competitive spot market in which houses are exchanged for consumption at a price of \( q_t \), while the other is a one-period credit market in which one unit of consumption at date t is exchanged for a claim to \( R_t \) units of consumption at date \( t + 1 \).

The expected utility of a household \( i \) is

\[
E_0^{P^i} \sum_{t=0}^{\infty} (\beta^B(i))^t c^B_t(i)
\]

where \( \beta^B(i) \) is his subjective discount factor and \( c^B_t(i) \) his consumption in period \( t \). The operator \( E_0^{P^i} \) denotes household \( i \)'s expectation in some probability space \( (\Omega, S, P^i) \), where \( \Omega \) is the space of payoff relevant outcomes that the household takes as given in his optimization problem. The probability measure \( P^i \) assigns probabilities to all Borel subsets \( S \) of \( \Omega \). It may or may not coincide with objective probabilities emerged in the equilibrium. Further details about \( \Omega \) and \( P^i \) will be provided in the next section.
The household $i$ produces with a constant return to scale technology. His production function is

$$y_{t+1}^B(i) = (a + \tau)H_t^B(i)$$

where $H_t^B(i)$ is the amount of used houses. Only the $aH_t^B(i)$ component of the output is tradable in the market, while $\tau H_t^B(i)$ is perishable and non-tradable. The introduction of non-tradable output is to avoid continual postponement of consumption by households.

The household’s production technology is assumed to be idiosyncratic in the sense that it requires his specific labor input. He always has the freedom to withdraw his labor, or in the language of Hart and Moore (1994), the household’s human capital is inalienable. The households are potentially credit-constrained. The financial intermediaries protect themselves against risks of default by collateralizing the households’ houses. The household $i$ can at most pledge collateral

$$E_t^{P_i} q_{t+1} H_t^B(i)$$

Thus his borrowing constraint is

$$b_t^B(i) \leq (1 - \tau) \frac{E_t^{P_i} q_{t+1}}{R_t} H_t^B(i)$$

where $b_t^B(i)$ is the amount of loans, $E_t^{P_i} q_{t+1}$ the financial intermediary $j$’s expectation about the collateral price in period $t + 1$, and $R_t$ gross interest rate between $t$ and $t + 1$. The borrowing constraint says that a household can get a maximum loan which is equal to a fraction of the discounted and expected liquidation value of his house holdings at $t + 1$.

The household faces a flow-of-fund constraint

$$q_t(H_t^B(i) - H_{t-1}^B(i)) + R_{t-1} b_{t-1}^B(i) + c_t^B(i) \leq y_t^B(i) + b_t^B(i)$$

He produces consumption goods using houses and borrows from the credit market. He spends on consuming, repaying the debt, and investing in houses.

A financial intermediary $j$’s preferences are specified by a linear utility function. She maximizes the following expected utility

$$E_0^{P_i} \sum_{t=0}^{\infty} (\beta^L(j))^t A_t c_t^L(j)$$

where $P^{ij}$ is her subjective probability measure and $\beta^L(j)$ is her subjective discount factor. $A_t$ is an i.i.d innovation to the financial intermediary’s patience factor with a bounded support $[A, \overline{A}]$ and $E[A_t] = 1$. She faces the following budget constraint:

$$q_t(H_t^L(j) - H_{t-1}^L(j)) + b_t^L(j) + c_t^L(j) \leq y_t^L(j) + R_{t-1} b_{t-1}^L(j)$$

If borrowers repudiate their debt obligations, lenders can repossess borrowers’ collateral by paying a transaction cost proportional to the expected liquidation value of the collateral $\tau E_t^{P_i} q_{t+1} H_t^B(i)$. One explanation is that debt enforcement procedures in real world are significantly inefficient and some value is lost during such procedures, as documented by Djankov, Hart, McHersh and Shleifer (2008).
where $H^L_t(j) - H^L_{t-1}(j)$ is her investment in collateral holdings. She uses a decreasing return to scale technology to produce, i.e., $y^L_{t+1}(j) = G^L(H^L_t(j))$, where $G^L > 0$, $G^L'' < 0$.

The aggregate supply of the collateral is assumed to be fixed at $\overline{H}$. Later I will assume that all households (financial intermediaries) have the same subjective discount factor $\beta^B = \beta^B(i)$ for all $i$ ($\beta^L = \beta^L(j)$ for all $j$) and households are less patient than financial intermediaries, i.e., $\beta^B < \beta^L$.

### 2.2 Optimality and Market Clearing Conditions

Individual household $i$‘s optimal decisions with respect to consumption, borrowing and collateral demand are similar to those in the original KM paper. Since the return to investment in collateral holding is sufficiently high, he prefers to borrow up to the maximum, consume only the non-tradable part of his output and invest the rest in collateral holdings. His optimal consumption is

$$c^B_t(i) = \tau H^B_{t-1}(i)$$

and optimal borrowing

$$b^B_t(i) = (1 - \tau) \frac{E^P_t q_{t+1}}{R_t} H^B_t(i)$$

The household uses both his own resources and external borrowing to finance collateral holdings. Given that the household consumes only the non-tradable output, his net worth at the beginning of date $t$ contains the value of his tradable output $a H^B_{t-1}(i)$, and the current market value of the collateral held from the previous period $q_t H^B_{t-1}(i)$, net of the debt payment, $R_{t-1} b^B_{t-1}(i)$. The household $i$‘s demand on collateral can be derived from (2), (4), (7), and (8)

$$H^B_t(i) = \frac{1}{q_t - \frac{1}{R_t} E^P_t q_{t+1}} [(a + q_t) H^B_{t-1}(i) - R_{t-1} b^B_{t-1}(i)]$$

where $q_t - \frac{1}{R_t} E^P_t q_{t+1}$ is the down-payment required to buy a unit of house.

---

5 Consider a marginal unit of tradable consumption at date $t$. The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding and produce consumption goods. In the next period, he will consume the nontradable part of production and invest further the tradable part, and so forth. Similar to KM, an assumption, i.e., $\frac{\tau}{1 - \frac{R-1}{R}(1-\tau)} > \frac{1}{\beta^L}$, is made to ensure that the discounted sum of utility of investing it at date $t$ will exceed the utility of immediately consuming it (see Online Appendix G), which is 1. Assumption $\beta^B < \frac{\beta^L}{\lambda}$ ensures that the return to investment will also be larger than the alternative choice, saving it for one period and then investing. Hence the collateral constraint will always be binding.

The above argument is valid when the economy is in a neighborhood of the steady state under RE. Online Appendix G shows that the collateral constraint is binding in the quantitative analysis of the learning model.
Every period the household \( i \) inherits debt \( b_{t-1}^B(i) \) from the previous period\(^6\) where

\[
b_{t-1}^B(i) = (1 - \tau) \frac{E_{t-1}^{\rho_j} q_t}{R_{t-1}} H_{t-1}^B(i) \tag{10}
\]

His debt repayment \( R_{t-1} b_{t-1}^B(i) \) is influenced by the expectation of collateral price at period \( t \) formed at period \( t-1 \), i.e., \( E_{t-1}^{\rho_j} q_t \). After plugging (10) into (9), the collateral demand of the household \( i \) is derived as follows

\[
H_t^B(i) = \frac{1}{q_t - \frac{1}{R_t} E_t^{\rho_j} q_{t+1}} (a + q_t - (1 - \tau) E_{t-1}^{\rho_j} q_t) H_{t-1}^B(i) \tag{11}
\]

Note borrowers’ collateral demand is influenced by expectations at two successive periods, \( E_{t-1}^{\rho_j} q_t \) and \( E_{t}^{\rho_j} q_{t+1} \). The former comes from the inherited debt repayment, see equation (10). The dependence may give rise to interesting dynamics under learning, as analyzed later.

A financial intermediary \( j \) is not credit constrained and her demand for collateral is determined by the following optimality condition

\[
\frac{1}{R_t} G^{\rho_j} (H_t^{L}(j)) = q_t - \frac{1}{R_t} E_t^{\rho_j} q_{t+1} \tag{12}
\]

For a marginal unit of collateral, the financial intermediary could get \( \frac{1}{R_t} G^{\rho_j} (H_t^{L}(j)) \) by producing by herself. Alternatively, she can sell it, lend the proceeds to a household at rate \( R_t \), and buy it back in period \( t+1 \) at the expected price \( E_t^{\rho_j} q_{t+1} \), which gives her \( q_t - \frac{1}{R_t} E_t^{\rho_j} q_{t+1} \). At the optimum, the two strategies give identical payoff at period \( t \).

Note households are less patient than financial intermediaries because \( \beta^B < \frac{\beta^L}{A} \). In equilibrium the former will borrow from the latter and the rate of interest rate will always be equal to the financial intermediaries’ rate of time preference; that is

\[
R_t = \frac{A_t}{\beta^L} \tag{13}
\]

Assuming homogeneity among all borrowers and all lenders, symmetric equilibrium requires \( H_t^B = H_t^B(i) \), \( H_t^L = H_t^L(j) \), \( b_t^B = b_t^B(i) \), and \( b_t^L = b_t^L(j) \). Aggregation yields \( H_t^B = \int_0^1 H_t^B(i), \ H_t^L = \int_0^1 H_t^L(j), \ b_t^B = \int_0^1 b_t^B(i), \) and \( b_t^L = \int_0^1 b_t^L(j) \). Denote by \( y_t \) the aggregate output in period \( t \), which is the sum of the production by borrowers and lenders

\[
y_t = \int_0^1 y_t^B(i) + \int_0^1 y_t^L(j) \tag{14}
\]

\[
= (a + \tau) H_{t-1}^B + G(H_{t-1}^L) \tag{15}
\]

\(^6\)For the initial period (10) is assumed to hold, i.e., \( b_0^B(i) = (1 - \tau) \frac{E_0^{\rho_j} q_0}{R_0} H_0^B(i) \)
Market clearing implies $H_t^B + H_t^L = \overline{H}$ and $b_t^B = b_t^L$. In equilibrium, user cost of collateral, i.e., the opportunity cost of holding collateral for one more period, is

$$u_t^c = q_t - \frac{1}{R_t} E_t^{\mathbb{P}} q_{t+1}$$

and equals to the present value of the marginal product of collateral.

Due to zero net supply of loans and collateral assets, aggregate consumption $c_t$ will equal to aggregate output $y_t$. Since aggregate investment is automatically zero in the model, I introduce a fixed, exogenous amount of autonomous investment following Boz and Mendoza (2013). This captures the investment and government absorption in the data. So the GDP in the model is the sum of aggregate consumption and investment

$$GDP_t = c_t + I$$

(16)

Denote $(\text{Debt}/GDP)_t$ the household credit market debt/GDP ratio, which is calculated by

$$(\text{Debt}/GDP)_t = b_t^B / GDP_t$$

(17)

Online Appendix A provides details on the steady state and the log-linearization of the model.

3 Equilibrium with Imperfect Knowledge

In the rational expectations equilibrium, agents are endowed with knowledge about the equilibrium mapping from the history of collateral holdings and lenders’ preference shocks to collateral prices. Below I assume homogeneous expectations among all agents but relax the assumption that the homogeneity of agents is common knowledge, in particular, agents do not know other agents’ discount factors and beliefs about future collateral prices. Relaxation of the informational assumption leads to agents in the model being uncertain about the equilibrium mapping between collateral prices and fundamentals.

3.1 The Underlying Probability Space and the Internally Rational Expectation Equilibrium

I now describe the probability space $(\Omega, S, \mathbb{P})$. Following Adam and Marcet (2011), I extend the state space of outcomes to contain not only the sequence of fundamentals, i.e., borrowers’ collateral holdings and the shock to lenders’ patience factor, but also other pay-off relevant variables: collateral prices. Both borrowers and lenders view the process for $q_t, A_t$ and $H_t^B$ as external to their decision problem and the probability space over which they condition their choices is given by $\Omega = \Omega_q \times \Omega_A \times \Omega_{H^B}$ where $\Omega_X = \Pi_{t=0}^\infty R_+$ and $X \in \{q, A, H^B\}$. The probability space contains all possible sequences of
prices, lenders’ preference shocks and borrowers’ collateral holdings. Denote the set of all possible histories up to period $t$ by $\Omega^t = \Omega^t_q \times \Omega^t_A \times \Omega^t_{HB}$ and its typical element by $\omega^t \in \Omega^t$. The RE belief is nested as a special case in which the probability measure $P$ features a singularity in the joint density of prices and fundamentals. Since equilibrium pricing functions are assumed to be known to agents under RE, conditioning their choices on the collateral price process is redundant.

The agents are assumed to be “Internally Rational” as defined below, i.e., maximizing their expected utility under uncertainty, taking into account their constraints, and conditioning their choice variables over the history of all external variables. Their expectations about future external variables are evaluated based on their consistent set of subjective beliefs specified in the subsequent subsection, which is endowed to them at the outset.

**Definition 1 Internal Rationality**

a) A household $i$ is “Internally Rational” if he chooses $(b^B_t(i), H^B_t(i), c^B_t(i)) : \Omega^t \to R^3$ to maximize the expected utility (1) subject to the flow-of-fund constraint (4), the collateral constraint (3) and his production function, taking as given the probability measure $P^i$.

b) A financial intermediary $j$ is “Internally Rational” if she chooses $(b^L_t(j), H^L_t(j), c^L_t(j)) : \Omega^t \to R^3$ to maximize the expected utility (5) subject to the flow-of-fund constraint (6) and her production function, taking as given the probability measure $P^j$.

Note the internal rationality of agents is tied neither to any specific belief system nor to the learning behavior of agents. However, the belief system is usually specified with some near-rationality concept and it is natural to introduce learning behavior of agents.

Below I specify the equilibrium of the economy. Let $(\Omega_A, P_A)$ be a probability space over the space of histories of preference shocks $\Omega_A$. Denote $P_A$ the ‘objective’ probability measure for lenders’ preference shocks. Let $\omega_A \in \Omega_A$ denote a typical infinite history of lenders’ preference shocks.

**Definition 2 Internally Rational Expectations Equilibrium**

The Internally Rational Expectation Equilibrium (IREE) consists of a sequence of equilibrium price functions $\{q_t\}$, where $q_t : \Omega^t_A \to R^+$ for each $t$, contingent choices $(c^B_t(i), c^L_t(j), b^B_t(i), b^L_t(j), H^B_t(i), H^L_t(j)) : \Omega^t \to R^6$ and probability beliefs $P^i$ for each household $i$ and $P^j$ for each financial intermediary $j$, such that (1) all agents are internally rational, and (2) when agents evaluate $(c^B_t(i), c^L_t(j), b^B_t(i), b^L_t(j), H^B_t(i), H^L_t(j))$ at equilibrium prices, markets clear for all $t$ and all $\omega_A \in \Omega_A$ almost surely in $P_A$.

---

7This follows Adam and Marcet (2011).
In the IREE, expectations about collateral prices are formed based on agents’ subjective belief system, which are not necessarily equal to the ‘objective’ density. Collateral prices and borrowers’ collateral holdings are determined by equations (11), (12) and market clearing conditions after agents’ probability measures $P$ are specified.

### 3.2 Agents’ Belief System and Optimal Learning Behavior

I now describe agents’ probability measure $P$ and derive their optimal learning algorithm. Agents’ belief system is assumed to have the same functional form as the RE solution. They believe collateral prices and borrowers’ collateral holdings depend on past aggregate borrowers’ collateral holdings.\(^8\) It can be represented as following:\(^9\)

\[
\begin{align*}
\hat{q}_t &= \zeta^m + \zeta^p \hat{H}^B_{t-1} + \epsilon_t \\
\hat{H}^B_t &= \zeta^m + \zeta^p \hat{H}^B_{t-1} + \varrho_t
\end{align*}
\]

given $\hat{H}^B_0$, where

\[
\left( \begin{array}{c} \epsilon_t \\ \varrho_t \end{array} \right) \sim iiN \left( \begin{array}{c} 0 \\ 0 \end{array} ; \begin{array}{cc} \sigma^2 & 0 \\ 0 & \sigma^2 \end{array} \right) \tag{20}
\]

Unlike under rational expectations, they are assumed to be uncertain about the parameters and the shock precisions $(\zeta^m, \zeta^p, \frac{1}{\sigma^2}, \zeta^m, \zeta^p, \frac{1}{\sigma^2})$, which is a natural assumption given that internally rational agents cannot derive the equilibrium distribution of collateral prices. Note agents’ beliefs about $(\zeta^m, \zeta^p, \frac{1}{\sigma^2})$ do not matter for equilibrium outcomes because only one-step ahead expectations about collateral prices enter the equilibrium under internal rationality in the model. So I omit belief updating equations for $(\zeta^m, \zeta^p, \frac{1}{\sigma^2})$ for the rest of the paper.

Denote $K$ the precision of the innovation $\epsilon_t$, i.e., $K \equiv \frac{1}{\sigma^2}$. Agents’ uncertainty at time zero is summarized by a distribution

\[
(\zeta^m, \zeta^p, K) \sim f
\]

The prior distribution of unknown parameters is assumed to be a Normal-Gamma distribution as follows

\[
K \sim G(\gamma_0, d_0^{-2}) \tag{21}
\]

\[
(\zeta^m, \zeta^p) \mid K = k \sim N((\theta^m_0, \theta^p_0)', (\nu_0 k)^{-1}) \tag{22}
\]

\(^8\)The shock to lenders’ preferences is observable but not included in agents’ regression. Including it will generate a singularity in the regression if initial beliefs coincide with the rational expectations equilibrium given it is the only shock in the model.

\(^9\)This is analogous to learning the parameter linking prices and dividend in stock pricing models. Note the dividend here is the marginal product of lenders and a function of borrowers’ collateral holding. After log-linearization, the (percentage deviation of) dividend is just a constant multiple of the (percentage deviation of) the borrowers’ collateral holding.
The residual precision $K$ is distributed as a Gamma distribution, and conditional on the residual precision $K$ unknown parameters $(\zeta^m, \zeta^p)$ are jointly normally distributed. The deviation of this prior from the REE prior will vanish assuming agents’ initial beliefs are at the RE value $\theta = \bar{\theta} = (\bar{\theta}^m, \bar{\theta}^p)'$, and they have infinite confidence in their beliefs about the parameters, i.e., $\gamma_0 \to \infty$, and $\nu_0 \to \infty$.

For the sake of notational compactness, I denote $y_t$ and $x_t$ the collateral price $\tilde{q}_t$ and $(1, \hat{H}^B_{t-1})$ in the rest of this section, respectively. $\theta_t \equiv (\theta_t^m, \theta_t^p)$ stands for the posterior mean of $(\zeta^m, \zeta^p)$.

Given agents’ prior beliefs (21) and (22), optimal behavior implies that agents’ beliefs are updated by applying Bayes’ law to market outcomes. Online Appendix B shows that the posterior distribution of unknown parameters is given by

$$K|\omega^t \sim G(\gamma_t, d_t^{-2}) \quad (\zeta^m, \zeta^p)'|K = k; \omega^t \sim N((\theta_t^m, \theta_t^p)', (\nu_t k)^{-1})$$

where the parameters $(\theta_t^m, \theta_t^p, \nu_t, \gamma_t, d_t^{-2})$ evolve recursively as following

$$\theta_t = \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \quad (25)$$
$$\nu_t = \nu_{t-1} + x_t x_t' \quad (26)$$
$$\gamma_t = \gamma_{t-1} + \frac{1}{2} \quad (27)$$
$$d_t^{-2} = d_{t-1}^{-2} + \frac{1}{2} (y_t - x_t' \theta_{t-1})' (x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} (y_t - x_t' \theta_{t-1}) \quad (28)$$

To avoid simultaneity between agents’ beliefs and actual outcomes, I assume information on the data, i.e., prices and collateral holdings, is introduced with a delay in $\theta_t$. The following learning rule using lagged data is used

$$\theta_t = \theta_{t-1} + (x_{t-1} x_{t-1}' + \nu_{t-1})^{-1} x_{t-1} (y_{t-1} - x_{t-1}' \theta_{t-1}) \quad (29)$$
$$\nu_t = \nu_{t-1} + x_{t-1} x_{t-1}' \quad (30)$$

This timing convention is standard in the adaptive learning literature. Note while the forecast functions are predetermined, because beliefs are updated using lagged data, agents’ expectations are not predetermined as they depend on period $t$ information.

Equations (29) and (30) are equivalent to the following Recursive Least Square (RLS) learning algorithm

$$\theta_t = \theta_{t-1} + g_t S_t^{-1} x_{t-1} (y_{t-1} - x_{t-1}' \theta_{t-1}) \quad (31)$$
$$S_t = S_{t-1} + g_t (x_{t-1} x_{t-1}' - S_{t-1}) \quad (32)$$

when the initial parameter is set to $\nu_0 = NS_0$ and where $g_t = \frac{1}{t+N}$. Then it can be shown that for subsequent periods we have $\nu_t = (t+N)S_t$, for $\forall t \geq 1$. Therefore, $N$ in the above equations measures the precision of initial beliefs.

---

10A micro-founded belief system justifying this delay can be provided following Adam and Marcet (2010).
The term \( y_{t-1} - x'_{t-1} \theta_{t-1} \) in equation (31) is agents’ price forecast error at period \( t \). According to (31) and (32), a surprise in agents’ price expectation will induce a revision of their beliefs or the parameters linking prices and fundamentals.

As standard in the literature, the learning rule with a small and constant gain sequence \( g_t = g > 0 \) is used in the quantitative exercise in section 5.

\[
\theta_t = \theta_{t-1} + g S_t^{-1} x_{t-1} (y_{t-1} - x'_{t-1} \theta_{t-1}) \\
S_t = S_{t-1} + g (x_{t-1} x'_{t-1} - S_{t-1})
\]

(33)

(34)

A Bayesian micro-foundation for this learning algorithm is provided in Online Appendix C. Agents discount past observations and give relatively more importance to new data, keeping track of the structural changes in the economy.

In the next section the learning algorithm (31)-(32) is used to study the convergence condition of the learning process which is also useful to understand the convergence under a constant learning rule.

4 Understanding the Learning Model

Formal analysis of the dynamics of the learning model is conducted in this section: the convergence of the learning process under RLS learning and the transition dynamics of persistent belief adjustment and prices changes are considered. It closes with informal discussions of some related papers.

4.1 Convergence of the Learning Process

Will the learning process converge to the REE under RLS Learning and under which condition? This can be analyzed by applying standard techniques elaborated in Evans and Honkapohja (2001). Recall agents perceive prices to evolve according to (18), while their beliefs are updated following (31) and (32). The state variables of the learning algorithm are \( x_t = (1 \ H_t B_t)' \). Agents’ conditional expectations are \( E_{t-1}^P \hat{q}_t = \zeta_{t-1} x_{t-1} \) and \( E_{t}^P \hat{q}_{t+1} = \zeta_t' x_{t} \) where \( \zeta_t \equiv (c_t^m, c_t^p)' \).

Log-linearizing borrowers’ collateral demand (11) under symmetric equilibrium yields

\[
\hat{H}_t^B = \frac{(\hat{q}_t - (1 - \tau) E_{t-1}^P \hat{q}_t) - (\hat{q}_t - \frac{1 - \tau}{R} E_{t}^P \hat{q}_{t+1})}{1 - \frac{1 - \tau}{R}} + \hat{H}_{t-1}^B - \frac{\frac{1 - \tau}{R} \hat{A}_t}{(1 - \frac{1 - \tau}{R})}
\]

(35)

\( \hat{H}_t^B \) depends on expectations of two successive periods: \( E_{t-1}^P \hat{q}_t \) and \( E_{t}^P \hat{q}_{t+1} \). The former arises from debt repayments and the latter from down-payment.

\footnote{This equation is useful for understanding the impulse response functions and discussed further in section 5.1.}
collateral prices satisfy three expectation terms and hence three belief terms: one comes from the housing re-sale price (see equation (12)), the other two from the down-payment and debt repayment (see equation (11) or (35)). So the actual price collateral holdings elasticity, i.e., the user cost of collateral with respect to borrowers’ collateral holdings, is defined as

\[
\eta = \frac{d \log u^c(H_t^B)}{d \log H^B} |_{H^B = H^B} = \frac{d \log G'(H_t^B)}{d \log H_t^B} |_{H_t^B = H^B} \times \frac{H^B}{H_t^B - H^B}
\]

It is the product of the financial intermediaries’ marginal product of collateral and the ratio of the households’ collateral holdings to the financial intermediaries’ at the steady state.

Denote $\frac{1}{\eta}$ the steady state elasticity of user cost of collateral with respect to borrowers’ collateral holdings.\(^{12}\) Substituting the conditional expectations into the log-linearized version of equations (11) and (12) under the symmetric equilibrium, I get the actual law of motion (ALM) for collateral prices under learning

\[
\hat{q}_t = T_1(\zeta_{t-1}^m, \zeta_{t-1}^p + T_2(\zeta_{t-1}^m, \zeta_{t-1}^p) \hat{H}_t^{B} + T_3(\zeta_{t}^p) \hat{A}_t
\]

where $T_2(\zeta_{t-1}^p, \zeta_{t}^p) = \frac{(\frac{1}{\eta} \zeta_{t-1}^m + \frac{1}{\eta} \gamma)}{1 - \frac{1}{\eta} \zeta_{t-1}^m}$. $T_1(\zeta_{t-1}^m, \zeta_{t-1}^p)$ and $T_3(\zeta_{t}^p)$ are presented in Online Appendix A. The Minimum State Variables (MSV) RE solution for collateral prices satisfies $T_1(\zeta_{t-1}^m, \zeta_{t}^p) = \zeta_{t-1}^m = 0$ and $T_2(\zeta_{t-1}^m, \zeta_{t}^p) = \zeta_{t}^m$.\(^{13}\)

Collateral prices under learning depend on three expectation terms and hence three belief terms: one comes from the housing re-sale price (see equation (12)), the other two from the down-payment and debt repayment (see equation (11) or (35)). So the actual price collateral holdings elasticity, i.e., the user cost of collateral with respect to borrowers’ collateral holdings, is defined as

\[
\eta = \frac{d \log u^c(H_t^B)}{d \log H^B} |_{H^B = H^B} = \frac{d \log G'(H_t^B)}{d \log H_t^B} |_{H_t^B = H^B} \times \frac{H^B}{H_t^B - H^B}
\]

It is the product of the financial intermediaries’ marginal product of collateral and the ratio of the households’ collateral holdings to the financial intermediaries’ at the steady state.

\(^{12}\)The elasticity is defined as

\[
\frac{1}{\eta} = \frac{d \log u^c(H_t^B)}{d \log H^B} |_{H^B = H^B} = \frac{d \log G'(H_t^B)}{d \log H_t^B} |_{H_t^B = H^B} \times \frac{H^B}{H_t^B - H^B}
\]

\(^{13}\)The MSV RE solution for borrowers’ collateral holdings and collateral prices are an AR(1) process and ARMA(1,1) process, respectively and suppressed here.
The admissible parameter space $\Theta_1 \equiv \{ (\eta, R, \tau) | \eta > 0, R > 1, 0 \leq \tau < 1 \}$.

This implies that the steady state leverage ratio or loan-to-value ratio is in the interval $(0, \frac{1}{R}]$, the elasticity $\frac{1}{\eta}$ is positive, and the (net) interest rate is positive.

Local stability of the Minimum State Variable REE is determined by the stability of the following associated ordinary differential equations

$$
\frac{d\zeta^m}{d\tau} = T_1(\zeta^m, \zeta^p) - \zeta^m
$$

$$
\frac{d\zeta^p}{d\tau} = T_2(\zeta^p, \zeta^p) - \zeta^p
$$

The following condition establishes a sufficient condition for the E-stability of the Minimum State Variable RE equilibrium.

**Proposition 4**

The MSV RE equilibrium for the economy is Expectationally-stable (E-stable) for all admissible parameters in $\Theta_1$.

**Proof.** see Online Appendix D. ■

The following illustration may help to understand the E-stability condition. Fixing agents’ beliefs about $\zeta^m$ at the RE value 0 and the price elasticity $\zeta^p$ above the RE value $\zeta^p$, so agents have an excessively optimistic forecast function. Recall the three effects discussed above. On one hand, a high $\zeta^p$ influences negatively on collateral prices because agents hold excessive debt repayment. On the other hand, a high $\zeta^p$ impacts positively on collateral prices: via excessively optimistic one-step ahead forecast function/price and excessive credit limits relative to RE.

The E-stability result says exactly that the negative effect of the excessive debt repayment will dominate for all admissible parameterizations. This in turn leads to a low realization of the price elasticity which pushes agents’ belief downward. Therefore, the asymptotic local stability of the REE is achieved. Roughly speaking, given that the E-stability condition is satisfied and the parameter estimates are around the neighborhood of the RE value, we have $\beta_t \rightarrow \bar{\beta}$ and $\nu_t \rightarrow \infty$ almost surely.\textsuperscript{14}

Unlike the RLS learning algorithm, parameter estimates coming from a constant gain learning algorithm (33)-(34) can not point-converge to a single value even in a time-invariant economy, but they could still converge in distribution around the RE value as long as the gain parameter is sufficiently small.\textsuperscript{15}

\textsuperscript{14}Once the convergence of agents’ estimates in the collateral price process is achieved, agents’ belief about the parameter estimates in the borrowers’ collateral holding equation will also converge to the RE value.

\textsuperscript{15}The convergence properties of learning models under the constant-gain learning algorithm are discussed in details in Evans and Honkapohja (2001).
4.2 Transitional Learning Dynamics

Although eventually agents’ beliefs will converge to the REE beliefs under the RLS learning rule (29)-(30), the learning model may display strong persistence in belief and price changes during the transition to the REE. This is interesting given that house price changes display strong positive serial correlation at short time horizon, such as one year, as shown by Case and Shiller (1989), and Glaeser and Gyourko (2006).

Following Adam, Marcet and Nicolini (2012), one way to capture the strong persistence in the change of agents’ beliefs is the following definition of momentum. Momentum in belief adjustments is the key property for their asset pricing model replicating a number of equity pricing facts in the U.S. data, such as the return volatility, the persistence and volatility of price dividend ratio, etc.

Recall $p_t$ is agents’ belief about the price collateral holdings elasticity at period $t$, and $p$ the corresponding value at the RE level.

**Definition 5 Momentum**

Momentum is defined as:

1. if $\zeta_t^p > \zeta_{t-1}^p$ and $\zeta_t^p < \bar{\zeta}^p$, then $\zeta_{t+1}^p > \zeta_t^p$.
2. if $\zeta_t^p < \zeta_{t-1}^p$ and $\zeta_t^p \geq \bar{\zeta}^p$, then $\zeta_{t+1}^p < \zeta_t^p$.

The definition says following. Suppose agents’ belief or parameter estimate is adjusted upward (downward), i.e., $\zeta_t^p < \bar{\zeta}^p$ ($\zeta_t^p > \bar{\zeta}^p$), but still not exceed (not below) the RE level, i.e., $\zeta_t^p \leq \bar{\zeta}^p$ ($\zeta_t^p \geq \bar{\zeta}^p$), this will be followed by further upward (downward) belief adjustment, i.e., $\zeta_{t+1}^p > \zeta_t^p$ ($\zeta_{t+1}^p < \zeta_t^p$). Roughly speaking, agents’ optimism (pessimism) is followed by further optimism (pessimism).

To study the internal dynamics of the learning model, a deterministic model is examined by assuming $A_t = 0$ for all $t$. I further consider a simplified perceived law of motion without learning about $\zeta^m$ or the steady state, that is, $\widehat{q}_t = \zeta_{t-1}^p \widehat{H}_{t-1}^B + \omega_t$.

The T-map mapping from the subjective belief to the parameter in the actual law of motion is

$$T_2(\zeta_t^p, \zeta_t^p) = \left( \frac{1 - \frac{1}{T_1} + \frac{1}{T_1} \widehat{q}_t}{1 - \frac{\zeta_t^p}{\zeta_t^p} \zeta_t^p} \right)$$

which also determines critically the dynamics of the model with learning about $\zeta^m$. Below I characterize a sufficient condition for the learning model to display momentum in belief adjustments.

Substituting the actual law of motion for collateral prices, i.e., $q_t = T_2(\zeta_t^p, \zeta_t^p) \widehat{H}_t^B$, into agents’ belief updating equations (31)-(32) or (33)-(34) yields

$$\zeta_{t+1}^p = \zeta_t^p + g_t \left( \widehat{H}_t^B (\widehat{q}_t - \widehat{q}_t^p) \zeta_t^p \right)$$

where $g_t$ can be the decreasing gain sequence or constant gain and is positive. Assuming $\widehat{H}_t^B \neq 0$ below. Equation (37) says that agents will update their belief about the price...
elasticity upward (downward) if the realized price elasticity is higher (lower) than their subjective estimate.

Performing the first-order Taylor approximation of the $T_2$—map around the REE belief yields

$$T_2(\zeta_{t-1}^p, \zeta_t^p) \approx \zeta^p + \left(- \frac{\partial T_2}{\partial \zeta_{t-1}^p} \bigg|_{\zeta_t^p} \right) \left( \zeta^p - \zeta_{t-1}^p \right) - \left( \frac{\partial T_2}{\partial \zeta_t^p} \bigg|_{\zeta_t^p} \right) \left( \zeta^p - \zeta_t^p \right) \tag{38}$$

As explained in section 4.1, we know $\frac{\partial T_2}{\partial \zeta_{t-1}^p} \bigg|_{\zeta_t^p} < 0$ and $\frac{\partial T_2}{\partial \zeta_t^p} \bigg|_{\zeta_t^p} > 0$. A past belief $\zeta_{t-1}^p$ which is lower than the RE value implies a lower debt repayment (given collateral holdings) relative to that under RE and contributes positively to the realized price elasticity. A current belief $\zeta_t^p$ which is lower than the RE value contributes negatively to the price elasticity.

To illustrate, consider a scenario in which agents’ belief arrives at the RE value from below, i.e., $\zeta_{t-1}^p < \zeta_t^p = \zeta_t^p$. According to (38), the realized price elasticity will be larger than the RE value, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) > \zeta_t^p$. The past belief lower than RE value implies a lower debt repayment and helps to generate a high price elasticity. Furthermore, belief updating rule (37) implies agents will update their belief further upward, i.e., $\zeta_{t+1}^p > \zeta_t^p$. So there is a tendency that agents’ belief will overshoot the RE value when arriving at the RE value. This is true for any admissible parameterization of the model.

However, a stronger condition is needed for the learning model to display momentum in belief adjustments when agents’ belief is updated upward but still below the RE value, that is, $\zeta_{t-1}^p < \zeta_t^p < \zeta_t^p$. A sufficient condition to ensure momentum is

$$- \frac{\partial T_2}{\partial \zeta_{t-1}^p} \bigg|_{\zeta_t^p} \geq \frac{\partial T_2}{\partial \zeta_t^p} \bigg|_{\zeta_t^p} \tag{39}$$

The reason is as follows. Consider agents’ belief is updated upward but still below the RE value, that is, $\zeta_{t-1}^p < \zeta_t^p < \zeta_t^p$. If the above condition holds, (38) implies that the price collateral holdings elasticity in the ALM will be higher than the RE value, i.e., $T_2(\zeta_{t-1}^p, \zeta_t^p) > \zeta_t^p$. Using the realized price elasticity, agents will update their belief further upward according to (37).

Note if $\tau$ were 1 or equivalently the leverage ratio were zero, momentum would not arise in the learning model. This is because past beliefs would not appear in the $T_2$—map. (39) would not be met because its left hand side would be zero and its right hand side positive. There would not be an overshoot of agents’ belief because without a shock it would stay at the RE value when arriving there.

---

16 The magnitude of this further upward belief adjustment will depend on the size of the gain parameter. Also note the property that agents’ beliefs hover around the RE value during the learning transition does not conflict with the convergence of agents’ belief under the RLS learning to the RE value.

17 Note the set of parameters satisfying the momentum condition $\frac{\partial T_2}{\partial \zeta_{t-1}^p} \bigg|_{\zeta_t^p} + \frac{\partial T_2}{\partial \zeta_t^p} \bigg|_{\zeta_t^p} \leq 0$ is a subset of the set of parameters satisfying the E-stability condition $\frac{\partial T_2}{\partial \zeta_{t-1}^p} \bigg|_{\zeta_t^p} + \frac{\partial T_2}{\partial \zeta_t^p} \bigg|_{\zeta_t^p} < 1$. 

18
Figure 3: Threshold function $\frac{1-\tau}{R}(\frac{1}{\eta})$ and parameter combinations generating momentum

As we increase the leverage ratio above some threshold, (39) will be met so that the positive effect of a lower past belief or past debt repayment will dominate and the realized price elasticity will be sufficiently high and reinforce the initial optimism. This is shown by the following proposition which provides an explicit characterization of the momentum condition (39).

**Proposition 6**

A sufficient condition guaranteeing momentum in beliefs (around the neighborhood of the REE belief) in the learning model is that parameter combinations of $(\eta, R, \tau)$ satisfy $\frac{1-\tau}{R} \geq \frac{1}{g(R)\frac{1}{\eta}+1}$ where $g(R) = R(\sqrt{(R-1) + \frac{(R-1)^2}{4}} + \frac{R-1}{2})$.

**Proof.** see Online Appendix E. ■

As an example, I set the gross quarterly interest rate $R$ to 1.0088, which is the steady state value of the interest rate I choose in the quantitative exercise later. The threshold steady state loan-to-value ratio as a function of $\frac{1}{\eta}$, i.e., $\frac{1-\tau}{R} = \frac{1}{g(R)\frac{1}{\eta}+1}$, is plotted, which is decreasing in the elasticity $\frac{1}{\eta}$. The shaded area of figure 3, which is also the area above the threshold function, summarizes the parameter combinations $(\frac{1-\tau}{R}, \frac{1}{\eta})$ under which there is momentum in beliefs in the learning model. As can be

---

18 The parameter combinations generating momentum in beliefs are not sensitive to a wide range of the steady state value of the interest rate $R$ chosen here.
seen from this figure, momentum\textsuperscript{19} will arise in the learning model when the elasticity of the user cost with respect to borrowers’ collateral holdings is relatively large or the steady state leverage ratio is relatively large. Given the elasticity $\frac{1}{\eta}$, the leveraged economy with a sufficiently high steady state leverage ratio can display momentum in belief and price changes.

Online Appendix F provides a discussion of the robustness of the qualitative learning dynamics with respect to an alternative specification of the collateral constraint.

4.3 Discussions

4.3.1 Relation to Adam, Kuang, and Marcet (2011, henceforth AKM)

AKM develop an open economy asset pricing model with a housing collateral constraint and learning which quantitatively accounts for the heterogeneous G7 house prices and current account dynamics over 2001-2008. In the AKM model, the price boom-bust is the consequence of the dynamic interaction of only house prices and price beliefs. Feedback from credit expansion/contraction to house prices, which is believed to have played a critical role during the recent U.S. housing cycle, has been shut down in the AKM analysis.\textsuperscript{20} For example, studying detailed zip code level data, Mian and Sufi (2009) suggest that “there may be a feedback mechanism between credit growth and house price growth” and “the evidence cautions against treating house prices movements in the last decade as independent from the expansion and collapse of subprime mortgage securitization.” The current model features dynamic interaction of house prices, price beliefs and credit limits and can capture this important feedback.

Both AKM and the current model generate quantitatively significant differences from the RE version of the models. For both models, the critical property is the dependence of collateral prices on belief changes and hence the possibility of endogenously persistent belief and price changes. The improvement in the fit of the model here arises from the intrinsic property of a collateral-constrained economy that prices are influenced by past beliefs via debt repayment, contrast to from learning about the permanent component of house price growth in AKM.

Different from AKM, the paper also analyzes the role of leverage ratio, convergence of the learning process, and studies the dynamics of household debt and aggregate consumption quantitatively.

\textsuperscript{19}Though the parameterizations in the quantitative exercise later do not fall in the shaded area here, the persistence in agents’ beliefs and in collateral price changes can still arise when the learning friction interacts with interest rate reductions.

\textsuperscript{20}In AKM, price fluctuations affect the collateral values of domestic households, borrowing, and current account dynamics. Relaxation and tightening of credit limits have an effect on the borrowers’ housing demand but no impact on the house price dynamics in the approximate solution. The latter is because the marginal product of houses ($\xi_i G’(H_i)$) is kept constant in the quantitative analysis and hence the collateral demand function is horizontal. No feedback from credit limits to house prices is evident by inspecting the key equations of the learning model in AKM, i.e., the belief updating equation (33) and the actual law of motion for house prices (35).
4.3.2 Relation to some asset pricing models with adaptive learning

The results relate to some papers studying asset pricing models with adaptive learning in an endowment economy, e.g., Timmermann (1996) and in a production economy without collateral constraints, e.g., Carceles-Poveda and Giannitsarou (2008, henceforth CG). They find a limited role of adaptive learning in asset pricing when agents learn about the parameters linking asset prices to fundamentals or dividends. In particular CG follow the Euler-equation learning approach, replacing expectation terms in agents’ first-order conditions by those formed by adaptive learning. Preston (2005) shows that the decision rule leads to suboptimal decisions. In standard real business cycle models, Eusepi and Preston (2011) suggest that such learning approach are unlikely to be helpful in explaining quantitative features of macroeconomic dynamics.

In the current paper agents’ optimal decisions are fully characterized by Euler equations. This is due to the risk-neutrality assumption of agents’ preferences. Agents make fully optimal decisions conditional on beliefs, which is identical to the framework proposed by Preston (2005) and Eusepi and Preston (2011) up to the precise specification of beliefs. In Timmermann (1996) and CG, when agents’ beliefs arrive at the RE beliefs, they will stay there without further realizations of shocks. Therefore, momentum in belief changes and prices changes are not present.

The asset pricing equation in the collateral-constrained economy with learning differs critically from them. Collateral prices are directly influenced by past beliefs and hence by the change of agents’ price beliefs. This opens the possibility for the learning model to display strong persistence in belief changes and larger fluctuations of prices and quantities even if a similar belief specification as them is considered, i.e., agents learn about the parameters linking house prices and fundamentals or dividends.

5 Quantitative Results and Further Mechanism

The learning model is estimated to the U.S. economy showing that the learning model can quantitatively account for the recent house prices boom and bust and the associated household debt and aggregate consumption dynamics. Around the year 2001, the U.S. real interest rate dropped considerably and stayed low for an extended period of time, before rising again around the year 2006. The average of 1-year ahead ex-ante real mortgage interest rates\(^ {21}\) from 1997Q1 to 2000Q4 was 3.51%, while the average of real interest rates between 2001Q1-2005Q4 was 2.28%.

The following experiment is conducted. Initially the economy is assumed to be at the steady state and agents’ beliefs at 2000Q4 are set to the RE value. The low real interest rates after 2000Q4 and the subsequent increase are captured in the following stylized way. The annualized real interest rate at the steady state is set to 3.51%. I let

\(^ {21}\)The mortgage rate used is the “one-year adjustable rate mortgage average in the United States” from Freddie Mac (seriesID: MORTGAGE1US). The ex-ante real interest rate is calculated as the mortgage rate minus the median expected 1 year ahead CPI inflation rate from the survey of professional forecasters.
the interest rate fall from 2001Q1, stay unchanged at 2.28% until 2005Q4, and then go back to the steady state. The model is used to generate real house prices, consumption and debt/GDP ratio during 2001Q1-2008Q4. Following Campbell (1994), I set the steady state consumption-GDP ratio to 0.745.

Denote by $c_k$ the product of the productivity gap $\frac{(a+\tau)-G'}{(a+\tau)}$ and borrowers’ production share $\frac{a+e}{y}$ in aggregate output. The gain parameter $g$, the elasticity $\frac{1}{\beta}$, the parameter $\tau$, and the parameter $ck$, are chosen to minimize the absolute distance between the learning model generated and actual house prices, consumption and debt/GDP ratio as follows

$$\sum_{t=2001Q1}^{2008Q4} \left( \frac{|\hat{q}_t - \bar{q}_t|}{std(\bar{q}_t)} + \frac{|\hat{c}_t - \bar{c}_t|}{std(\bar{c}_t)} + \frac{|Debt/GDP_t - Debt/GDP_t|}{std(Debt/GDP_t)} \right)$$

where boldface letters denote actual data and std stands for standard deviation.

The minimization yields that $g = 0.065$, $\frac{1}{\beta} = 2.46$, $\tau = 0.45$, and $ck = 0.43$. This choice of $\tau$ implies that the steady state loan-to-value ratio is 0.54. The value of $ck$ implies roughly, say both the productivity gap and borrowers’ production share are $\frac{2}{3}$. The choice of the parameters yields impulse response functions of the learning model broadly consistent with other studies. The parameterization of the RE models is the same as the estimated learning model.

Recall the interest rate at period $t$ in my model is $R_t = A_t L_t$. To get low interest rates during 2001-2005, I assume lenders’ discount factors $L_t$ shift upward exogenously during 2001-2005 and back to their old value during 2006-2008.

Under RE, two alternatives are considered: the interest rate movement is either unanticipated or anticipated. Model predictions for the two alternatives are provided below. “RE-I” model refers to the RE model with unexpected interest rate reductions, while “RE-II” model stands for the RE model with anticipated interest rates movement. Note the performance of the learning model will be the same under either of the two assumptions because expectations about future interest rates do not enter the system of equations governing the model economy.

5.1 Response to 1% unexpected interest rate reduction

Figure 4 depicts the responses to an unexpected interest rate reduction. In the impact period, real house price under RE rises by about 1.2%, while consumption and

---

22This is consistent with the estimate of the household loan-to-value ratio by Iacoviello (2005) with mean 0.55 and standard deviation 0.09.

23The productivity gap of $\frac{2}{3}$ is also considered by Cordoba and Ripoll (2004) in their figure 5.

24Iacoviello and Neri (2010) find that 1% positive i.i.d. monetary policy shock leads to a decrease of house prices by about 0.65% and hump-shaped response of consumption with the trough 0.5% below the steady state.

25Admittedly, this is a short-cut, but necessary way, to model the interest rate reduction in my context.
debt/GDP ratio rise by about 0.6% and 2%, respectively. However, they do not rise further after the interest rate reduction disappears. Consumption decays exponentially, while the house price drops substantially due to the disappearance of the interest rate reduction and then converges persistently to the steady state.

Under rational expectations, borrowers’ demand for collateral increases following an unexpected interest rate reduction. In the impact period, collateral is transferred from lenders to borrowers. Due to the fixed supply of collateral and the decreasing return to scale technology of lenders to produce, user cost of collateral rises above its steady state value. Since borrowers’ current investment in collateral holding raises their ability to borrow in the next period, there will be persistence in their collateral holdings. The user cost of collateral stays above the steady state for many periods. Under RE, the collateral price is the discounted sum of current and future user cost. The persistence in user cost reinforces the effect on collateral prices and collateral values, which leads to a larger effect on collateral transfers and aggregate activities.

After the disappearance of the interest rate reduction, the user cost rises above the steady state, which chokes off further rises in borrowers’ demand for collateral. Collateral prices and borrowers’ collateral holdings will revert immediately toward the steady state. Prices and quantities converge persistently and monotonically to the steady state.

The learning model generates additional propagation of the interest rate reduction due to belief revisions and the dynamic interaction between beliefs and price realizations. The peak responses of house prices, consumption and debt/GDP ratio are 1.2%, 0.73%, and 2.3%, respectively. The learning model also generates positive persistence in forecast errors, as can be seen from the lower right panel.

Under learning, the impact responses are the same as those under RE because agents have correct forecast functions initially. After the disappearance of the interest rate reduction, a positive surprise in the collateral price induces an upward belief revision. Agents partially interpret the surprise in collateral price forecasts as a permanent change in the parameters governing the collateral price process. They become more optimistic about future collateral prices due to both more optimistic beliefs and rising amounts of collateral holdings by borrowers. Credit limit is relaxed based on the optimism. As can be seen from equation (35), borrowers’ collateral holdings can rise if \( \frac{1-\tau}{\bar{R}} E_t P \hat{q}_{t+1} > (1 - \tau) E_{t-1} P \hat{q}_t\). With a larger borrowing capacity, borrowers can repay the debt and increase investment in collateral holdings.

Since aggregate consumption is a constant fraction of borrowers’ collateral holdings, consumption follows closely borrowers’ collateral holdings. The debt/GDP ratio rises due to rising collateral values which in turn is the consequence of both the price increase and rising amounts of borrowers’ collateral holdings.

In the period 2, collateral price under learning is much higher than that under

\[26\] Note the debt/GDP ratio here is percentage changes from the steady state value.

\[27\] The forecast error is defined as \( \hat{q}_t - E_{t-1} \hat{q}_t \).

\[28\] See Online Appendix A for the analytical expressions for log-linearized consumption and debt/GDP ratio.
Figure 4: Response to 1% unexpected negative shock to interest rate

RE mainly due to more optimistic expectation about future prices. The temporary decline in collateral prices is due to the return of the interest rate to the steady state. Nevertheless, the realized price is still higher than agents’ price forecast. So the realized price reinforces agents’ optimism, which leads to further optimism when the price realization is used for belief updating. The positive effects of more optimistic beliefs and expectations can temporarily dominate the negative effect of debt repayment and lead to a rise in collateral price. Learning about collateral prices gives rise to the dynamic feedback between agents’ beliefs and actual prices through the relaxation of credit limits, which generates prolonged periods of expansion of prices and quantities.

The reversal of prices and quantities relates to the convergence of the learning process or the Expectational-stability analysis of the REE in section 4.1. At some point, the debt repayment becomes excessive such that its negative effect dominates and the realized collateral prices fall short of expectations. This sets a self-reinforcing decline in motion. According to (33) and (34), agents’ beliefs are revised downward and they become pessimistic. Credit limits are tightened based on the pessimism. Borrowers’ collateral demand falls, so does the realization of collateral prices. The realized collateral prices reinforce agents’ initial pessimism, which generates further declines in prices and quantities. Eventually prices and quantities converge to the steady state.
5.2 Boom and bust in house prices, debt and aggregate consumption dynamics

Figure 5 contrasts model predictions of the learning model and of the “RE-I model” with actual data. Under RE, prices and quantities increase above their steady state values following the real interest rate reduction. House prices continue to increase due to the persistence in the user cost and the persistently low interest rates. They peak at about 14.4% above the steady state. After the disappearance of the interest rate reduction, house price starts to revert to their steady state. The RE model underpredicts considerably the levels of prices and quantities. 29

The learning model predicts house prices, debt/GDP ratio and consumption rather well, in particular during the price boom years. Following the real interest rate reduction, real house prices under learning increase at a faster pace than under RE. The learning model generates large additional amplification of prices and quantities relative to the RE version of the model. The peak of the predicted house prices under learning is about 35.9% at 2006Q4, which is about 2.5 times the peak response of under RE.

29Given the pattern of the interest rates I consider, the response of house prices in the “RE-I” model will be larger if the elasticity is larger. The improvement of the performance of the RE model with a larger or a larger leverage ratio is limited when the leverage is not very large or close to 1 and the response of consumption or the collateral holding transfer is not counterfactually large. Regardless of the value of these two parameters, the REE house prices will revert when the interest rate starts to revert. So the RE model cannot match the turning point of house prices.
The house price boom arises mainly from more optimistic expectation about future prices due to both more optimistic beliefs and the rising amount of collateral held by households. The rising household credit market debt/GDP ratio is due to both the house price boom and the rising amount of collateral held by households. The learning model also generates a consumption boom due to shifts of collateral to more productive households. The peak response of consumption in the learning model is 18.8%, which is twice as large as that in the RE model.

House prices in the “RE-I model” start to revert once the interest rate rises, while the learning model matches rather well the turning point of house prices in the data. House prices in the learning model rise further for a few quarters as in the data even after the rise of the interest rates. This is due to belief revisions and the interaction of beliefs and price realizations.

The forecast errors of the RE model are constant during 2001Q1-2005Q4 and then become zero afterwards. They are completely driven by and correspond to the pattern of exogenous shifts in interest rates. In contrast, the learning model generates internal and positive persistence in forecast errors. Gelain, Lansing, and Mendicino (2013) derived house price forecasts from the future markets for the Case-Shiller house price index (where only the data from 2006 onwards are available) and showed that “the future market tends to overpredict future house prices when prices are falling.” The learning model generates exactly this pattern of forecast errors as one can see from the lower right panel of figure 5 that the real house price forecast error becomes negative from 2006 onwards.

Similar to the KM model, assumptions have been made to ensure that the return to investment is higher than that to consumption and saving in a neighborhood of the steady state, so that the collateral constraint is always binding under RE. In the learning model agents evaluate the payoffs of different strategies using the subjective probability measure and there may be substantial deviations of beliefs and expectations. It is unclear that if the collateral constraint remains binding in this case without a formal check. Online Appendix G confirms that the collateral constraint is indeed binding during the housing cycle over the year 2001-2008 in the learning model.

5.3 RE price dynamics with anticipated interest rate movement

Figure 6 displays the “RE-II” model dynamics, i.e., when the low interest rates during 2001Q1-2005Q4 are anticipated by the agents. Except for the initial period, agents understand the effects of such structural change and could perfectly foresee the entire path of prices and quantities given that there is no remaining uncertainty after the initial real rate reduction. The real house prices jump immediately upward and then converge to the steady state. This is inconsistent with the pattern of prices and

---

30 For simulating the model in such scenario, I firstly solve the law of motion for prices and quantities during 2006Q1-2008Q4. Then with them I recursively solve backward the policy function until 2001Q1.
quantities observed in the data. In particular, the model does not generate persistent increases in house prices.

6 Conclusions

The paper presents a general equilibrium model with a housing collateral constraint and learning which can quantitatively account for the recent U.S. boom and bust in house prices, as well as the household debt and aggregate consumption dynamics following the persistent fall in real interest rates after the year 2000. Agents are uncertain about the parameters linking prices and fundamentals and their beliefs feature small deviation from REE beliefs. The role of the collateral constraint as an amplification mechanism is significantly strengthened via the dynamic interaction of agents’ price beliefs, price realizations, and credit limits. For example, more optimistic (pessimistic) beliefs leads to enhanced (tightened) borrowing capacity, which boosts (dampens) collateral demand and collateral prices. The price realizations partially validate agents’ optimism (pessimism) and lead to further optimism (pessimism). It is shown that the leveraged economy with a higher leverage ratio is more prone to such self-reinforcing learning dynamics.

The nonlinear dependence of economic volatilities on the leverage ratio may help to understand economic volatilities of aggregate variables across regimes with different leverage ratios or in cross-country comparisons. For example, in studying the behavior of money, credit and macroeconomic indicators for 14 countries over the year 1870-
2008, Schularick and Taylor (2011) find output losses today are as large as Pre-WW2 despite more activist policies and the presence of deposit insurance and allude to the important role of increased leverage in the financial sector. Second, the model provides an additional rationale for reasonable capital requirement regulation to avoid an extremely high leverage ratio regime which is more prone to self-reinforcing learning dynamics and to generate excess volatilities of prices and quantities.

Asset prices/values play a large role in aggregate fluctuations through many channels such as households, corporate balance sheets, bank capital channels, etc. It would be interesting to study and quantify further the role of the interaction of agents’ uncertainty in financial markets considered here with other kind of credit market frictions in aggregate fluctuations. It would also be interesting to look into how the uncertainty in financial markets interacts with economic agents’ decisions in other markets, such as the labor market. Finally, the model facilitates the discussion of how monetary policies can influence whether a bubble occurs in the first place and how they can affect the speed at which it deflates, as well as the appropriate design of policies to stabilize the economy and the financial system.

Acknowledgements

This paper is based on Chapter 2 of my dissertation. I am indebted to my supervisors Thomas Laubach and Klaus Adam for invaluable advice and numerous discussions. I am grateful to the Editor (Eric Leeper), an anonymous Associate Editor and a referee for helpful suggestions and comments. Thanks to Tobias Adrian, Roel Beetsma, Gianni De Fraja, George Evans, John Fender, Cars Hommes, Alex Ilek, Leo Kaas, Kevin Lansing, Kaushik Mitra, Kalin Nikolov, Bruce Preston, Olaf Posch, Ansgar Rannenberg, Sigrid Roehrs, Christian Schlag, Ctirad Slavik, Sergey Slobodyan, Jan Tuinstra, Raf Wouters and participants at 2013 San Francisco Fed Conference “Expectations in Dynamic Macro Models”, 2013 Workshop “Macroeconomic Policy and Expectations” (St Andrews), 3rd Bundesbank-CFS-ECB workshop on Macro and Finance (Frankfurt), 2010 conference “Expectation, Asset Bubbles, and Financial Crisis” (Rotterdam), SNDE 2011 (Washington DC), SMYE 2011 (Groningen), EEA 2011 (Oslo), University of Frankfurt, Mannheim, St. Andrews, Konstanz, City U Hong Kong, and SHUFE for helpful discussions and comments. Thanks to Emine Boz and Enrique Mendoza for providing part of the data. The financial support from the German Research Foundation (DFG) is gratefully acknowledged.

References


Online Appendix for “Imperfect Knowledge about Asset Prices and Credit Cycles” (Pei Kuang)

A Steady State, Log-linearization and the Actual Law of Motion under Learning

The steady state of the interest rate, user cost of collateral, collateral prices, lenders’ collateral holdings, borrowers’ collateral holdings, borrowing, and borrowers’ consumption are

\[ R = \frac{1}{\bar{p}} \], \[ u = \frac{a}{(1-\tau)} \], \[ q = \frac{aR}{(R-1)(1-\tau)} \], \[ H^L = C' \left( \frac{aR}{1-\tau} \right) \], \[ H^B = \bar{H} - H^L \], \[ b^B = (1-\tau)qH^B/R \] and \[ c^B = \bar{e}H^B \]. Recall the model equations

\[ H^B_t = \frac{a + q_t - (1-\tau)E_{t-1}^P q_t}{q_t - \frac{1}{R_t}(1-\tau)E_t^P q_{t+1}} H^B_{t-1} \] (40)

\[ q_t - \frac{1}{R_t} E_t^P q_{t+1} = \frac{G'(H^L_t)}{R_t} \] (41)

Equation (40) implies

\[
\begin{bmatrix}
q_t - \frac{1}{R_t} (1-\tau)E_t^P q_{t+1}
\end{bmatrix}
\begin{bmatrix}
H^B_t
\end{bmatrix} = \begin{bmatrix}
[a + q_t - (1-\tau)E_{t-1}^P q_t]
\end{bmatrix} H^B_{t-1}

\[
\begin{bmatrix}
q_t R_t - (1-\tau)E_t^P q_{t+1}
\end{bmatrix}
\begin{bmatrix}
H^B_t
\end{bmatrix} = \begin{bmatrix}
[a + q_t - (1-\tau)E_{t-1}^P q_t]
\end{bmatrix} H^B_{t-1} R_t
\]

Note \( \frac{a + \tau q}{q} = 1 - \frac{1 - \tau}{R} \). Log-linearizing this equation yields

\[
q [R - (1-\tau)] H^B_t \hat{H}^B + H^B \left[ R q \left( \hat{q}_t + \hat{R}_t \right) - (1-\tau)qE_t^P \hat{q}_{t+1} \right]
\]

\[
= (a + \tau q) RH^B \left( \hat{R}_t + \hat{H}^B_{t-1} \right) + RH^B \left( q\hat{q}_t - (1-\tau)qE_{t-1}^P \hat{q}_t \right)
\]

Further simplification yields

\[
\left( 1 - \frac{1 - \tau}{R} \right) \hat{H}^B_t + \left[ (\hat{q}_t + \hat{R}_t) - \frac{1 - \tau}{R} E_t^P \hat{q}_{t+1} \right]
\]

\[
= \left( 1 - \frac{1 - \tau}{R} \right) \left( \hat{R}_t + \hat{H}^B_{t-1} \right) + (\hat{q}_t - (1-\tau)E_{t-1}^P \hat{q}_t)
\]

Recall \( R_t = \frac{A_t}{\bar{p}_t} \), so \( \hat{R}_t = \hat{A}_t \). This implies further
\[ \hat{H}_t^B = \frac{1}{(1 - \frac{1 - \tau}{R})} \left[ (\hat{q}_t - (1 - \tau)E_{t-1}^P \hat{q}_t) - \left( \hat{q}_t - \frac{1 - \tau}{R} E_t^P \hat{q}_{t+1} \right) \right] + \hat{H}_{t-1}^B - \frac{1 - \tau}{(1 - \frac{1 - \tau}{R})} \hat{A}_t \] (42)

Now start with equation (41)

\[
q_t - \frac{1}{R_t} E_t^P q_{t+1} = \frac{G'(H_t^L)}{R_t} 
\]

\[ \iff R_t q_t - E_t^P q_{t+1} = G'(H_t^L) \] (43)

Recall the deﬁnition of \( \frac{1}{\eta} = -\frac{G''(H^L) H_t^L}{G'(H^L)} \). Note log-linearizing \( H_t^B + H_t^L = \bar{\mathcal{H}} \) implies

\[ \hat{H}_t^L = -\frac{H_t^B}{R_t} \hat{H}_t^B. \]

Log-linearization of equation (43) leads to

\[
R q (\hat{q}_t + \hat{R}_t) - q E_t^P \hat{q}_{t+1} = G''(H_t^L) H_t^L \hat{H}_t^L 
\]

\[ \quad = -\frac{G''(H_t^L) H_t^L H_t^B}{G'(H_t^L)} G'(H_t^L) \hat{H}_t^B 
\]

\[ \quad = \frac{1}{\eta} G'(H_t^L) \hat{H}_t^B 
\]

\[ \quad = \frac{1}{\eta} q (R - 1) \hat{H}_t^B 
\]

In the last equation, the steady state relationship, \( G'(H^L) = q (R - 1) \), is used. Simpliﬁcation further yields

\[ \frac{1}{\eta} \hat{H}_t^B = \frac{R}{R - 1} \left( \hat{q}_t - \frac{1}{R} E_t^P \hat{q}_{t+1} + \hat{R}_t \right) \]

Given that \( \hat{R}_t = \hat{A}_t \), the above equation leads to

\[ \frac{1}{\eta} \hat{H}_t^B = \frac{R}{R - 1} \left( \hat{q}_t - \frac{1}{R} E_t^P \hat{q}_{t+1} + \hat{A}_t \right) \] (44)

Below the actual law of motion (ALM) for collateral prices is derived. Suppose the perceived law of motion for collateral prices is \( \hat{q}_t = \zeta_t^m + \zeta_t^B \hat{H}_{t-1}^B + \zeta_t^A \hat{A}_t \). Conditional expectations at period \( t \) and \( t - 1 \) for collateral prices are \( E_t^P \hat{q}_{t+1} = \zeta_t^m + \zeta_t^B \hat{H}_t^B \) and \( E_{t-1}^P \hat{q}_t = \zeta_{t-1}^m + \zeta_{t-1}^B \hat{H}_{t-1}^B \). Equation (42) can be simpliﬁed to

\[ \hat{H}_t^B = \frac{1}{(1 - \frac{1 - \tau}{R})} \left[ 1 - \frac{\tau}{R} E_t^P \hat{q}_{t+1} - (1 - \tau)E_{t-1}^P \hat{q}_t \right] + \hat{H}_{t-1}^B - \frac{1 - \tau}{(1 - \frac{1 - \tau}{R})} \hat{A}_t \] (45)
Combining (44) and (45) by eliminating \( \hat{H}_t^B \), I get

\[
\hat{q}_t = \xi_1 E_t^p \hat{q}_{t+1} - \xi_2 E_{t-1}^p \hat{q}_t + \xi_3 \hat{H}_{t-1}^B + \xi_4 \hat{A}_t
\]

where \( \xi_1 = \frac{1}{R} + \frac{1}{\eta} \frac{R-1}{R} \frac{1}{\eta/(1-\gamma)} \frac{1-\tau}{1-\eta/(1-\gamma)} \), \( \xi_2 = (1-\tau) \frac{1}{\eta} \frac{R-1}{R} \frac{1}{\eta/(1-\gamma)} \), \( \xi_3 = \frac{1}{\eta} \frac{R-1}{R} \) and \( \xi_4 = -\frac{1}{R} \). Substituting conditional expectations into the last equation, I get

\[
\hat{q}_t = \xi_1 \left( \zeta_t^m + \zeta_t^p \hat{H}_t^B \right) - \xi_2 \left( \zeta_{t-1}^m + \zeta_{t-1}^p \hat{H}_{t-1}^B \right) + \xi_3 \hat{H}_{t-1}^B + \xi_4 \hat{A}_t
\] (46)

Substituting conditional expectations into (44), I get

\[
\frac{1}{\eta} \hat{H}_t^B = \frac{R}{R-1} \left( \hat{q}_t - \frac{1}{R} \left( \zeta_t^m + \zeta_t^p \hat{H}_t^B \right) + \hat{A}_t \right)
\]

or alternatively

\[
\hat{H}_t^B = \frac{\hat{q}_t + \hat{A}_t - \frac{1}{R} \zeta_t^m}{\frac{1}{\eta} \frac{R-1}{R} + \frac{1}{R} \zeta_t^p}
\] (47)

Combining (47) and (46) by eliminating \( \hat{H}_t^B \), I get the ALM for collateral prices under learning

\[
\hat{q}_t = T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) + T_2(\zeta_{t-1}^p, \zeta_t^p) \hat{H}_{t-1}^B + T_3(\zeta_t^p) \hat{A}_t
\]

where \( T_1(\zeta_{t-1}^m, \zeta_t^m, \zeta_t^p) = \frac{(\xi_1 \zeta_t^m - \xi_2 \zeta_{t-1}^m)(1+\zeta_t^m R) - \xi_3 \zeta_t^p}{1+\xi_3 R - \xi_1 \xi_3 \frac{1}{\eta} \frac{R-1}{R}} \), \( T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3 \zeta_{t-1}^p \frac{1}{\eta} \frac{R-1}{R}}{1+\xi_3 \frac{1}{\eta} \frac{R-1}{R}} \) and \( T_3(\zeta_t^p) = \frac{(\xi_4 + \xi_2 \zeta_t^p)}{1+\xi_3 \frac{1}{\eta} \frac{R-1}{R}} \). Those expressions are used for proving the propositions and computations.

### A.1 An alternative expression of the \( T_2 \)-map

In the paper, an alternative expression of the \( T_2 \)-map, i.e., \( T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\left( \frac{1}{\eta} \left( \frac{R-1}{R} \right) \left( \frac{1}{1-\frac{R-1}{1-\gamma}} \right) \zeta_{t-1}^p \right)}{1+\zeta_t^p \left( \frac{R-1}{R} \right) \left( \frac{1}{1-\frac{R-1}{1-\gamma}} \right) \zeta_t^p} \), is presented and used to illustrate some intuitions. Its derivation is provided here. I am only interested in deriving the \( T_2 \)-map and keeping track of agents’ beliefs, so I consider a simpler case without learning about \( \zeta_t^m \) and without shock \( \hat{A}_t \) for the simplicity of expositions. The \( T_2 \)-map in the full model with learning about the steady state and with stochastic innovations will be the same as that in the simpler case.

Recall our model equations are (44) and (45) which contain three expectation terms. Substituting conditional expectations into equation (45), we get
\[ \hat{H}^B_t = \frac{1}{R} \left[ \frac{1 - \tau}{R} E_{t}^{\rho} \hat{q}_{t+1} - (1 - \tau) E_{t-1}^{\rho} \hat{q}_{t} \right] + \hat{H}^B_{t-1} \]
\[ = \frac{1}{R} \left[ \frac{1 - \tau}{R} \zeta_t^p \hat{H}^B_t - (1 - \tau) \zeta_{t-1}^p \hat{H}^B_{t-1} \right] + \hat{H}^B_{t-1} \]

Note in the above equation \( \zeta_t^p \) comes from the down-payment and \( \zeta_{t-1}^p \) debt repayment. So

\[ \hat{H}^B_t = \frac{1 - \frac{(1 - \tau)}{\eta} \zeta_{t-1}^p}{1 - \frac{(1 - \tau)}{\eta} \zeta_t^p} \hat{H}^B_{t-1} \quad (48) \]

Substituting conditional expectations into equation (44)

\[ \hat{q}_t = \frac{1}{R} E_{t}^{\rho} \hat{q}_{t+1} + \frac{1}{R} \hat{H}^B_t \]
\[ = \frac{1}{R} \zeta_t^p \hat{H}^B_t + \frac{1}{\eta} \hat{H}^B_t \]
\[ = \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \right) \hat{H}^B_t \quad (49) \]

Note \( \zeta_t^p \) in the above equation arise from the housing re-sale price/value. Combining equation (48) and (49) yields the \( T_2 \)-map used in the paper

\[ \hat{q}_t = \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \right) \frac{1 - \frac{(1 - \tau)}{\eta} \zeta_{t-1}^p}{1 - \frac{(1 - \tau)}{\eta} \zeta_t^p} \hat{H}^B_{t-1} \]

From this equation, the three belief terms can be easily tracked. It can be shown that it is identical to the \( T_2 \)-map derived earlier, i.e., \( T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3 - \xi_2 \zeta_{t-1}^p}{\xi_3 + \xi_1 \zeta_t^p} \).

### A.2 Consumption and Debt/GDP ratio

Denote by \( Y \) the steady state value of aggregate output. Log-linearizing aggregate production yields

\[ \hat{y}_t = \frac{(a + \tau) - G' (a + \tau) H^B}{(a + \tau) Y} \hat{H}^B_{t-1} \]

It equals to the product of the productivity gap \( \frac{(a + \tau) - G'}{(a + \tau)} \) between borrowers and lenders, the production share of borrowers \( \frac{(a + \tau) H^B}{Y} \) and the redistribution of collateral. Aggregate consumption \( \hat{c}_t \) will be the same as aggregate output because of zero net investment in housing.
Denote by $\overline{C}$ and $\overline{GDP}$ aggregate consumption and GDP at the steady state, respectively. The log-linearized GDP is

$$\overline{GDP}_t = \frac{\overline{C}}{\overline{GDP}} \bar{c}_t$$

and the debt/GDP ratio is

$$(\text{Debt}/GDP)_t = \hat{b}^B_t - \overline{GDP}_t$$

$$(\text{Debt}/GDP)_t = E^P_{t+1} \hat{y}_{t+1} + H^B_t - \hat{R}_t - \frac{\overline{C}}{\overline{GDP}} \bar{y}_t$$

where $\hat{b}^B_t$ can be calculated by the collateral constraint.

\section*{B Derivation of the Bayesian Posterior Mean\textsuperscript{31}}

I assume the prior distribution of unknown parameters, i.e., the parameters linking prices and fundamentals $(\zeta^m, \zeta^p)$ and the residual precision $K \equiv \frac{1}{\sigma^2}$, is a Normal-Gamma distribution as following

$$K \sim G(\gamma_0, d_0^{-2})$$

$$(\zeta^m, \zeta^p) | K = k \sim N((\theta^m_0, \theta^p_0)', (\nu_0 k)^{-1})$$

The prior distribution of $K$ is a gamma distribution and the conditional prior of $\zeta \equiv (\zeta^m, \zeta^p)$ given $K$ is a multivariate normal distribution.

I drop the terms which do not involve $(\theta, k)$ by using the proportionality symbol. The conditional probability of the collateral price is a normal distribution with following conditional probability density function

$$p(y_t | \theta, k) \propto k^{\frac{1}{2}} \exp \left\{ -\frac{k}{2} (y_t - x'_t \theta)'(y_t - x'_t \theta) \right\}$$

The prior density of the parameters is following

$$p(\theta, h) \propto k^{\gamma_t-1} \exp \left\{ -d_{t-1}^{-2} k \right\} k^{\frac{1}{2}} \exp \left\{ -\frac{k}{2} (\theta - \theta_{t-1})'(\nu_{t-1}(\theta - \theta_{t-1})) \right\}$$

I proceed to show that the posterior distribution of the parameters is as follows

$$\theta | K = k \sim N(\theta_t, (\nu_t k)^{-1})$$

$$K \sim G(\gamma_t, d_t^{-2})$$

\textsuperscript{31}The derivation follows DeGroot (1974).
with probability density function
\[ p(\theta, k|y_t) \propto k^r \exp\{-d_t^{-2}k\} k^\frac{1}{2} \exp\{-\frac{k}{2}(\theta - \theta_t)'\nu_t(\theta - \theta_t)\} \]

where
\[ \theta_t = \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \]
\[ \nu_t = \nu_{t-1} + x_t x_t' \]
\[ \gamma_t = \gamma_{t-1} + \frac{1}{2} \]
\[ d_t^{-2} = d_{t-1}^{-2} + \frac{1}{2}(y_t - x_t' \theta_{t-1})' (x_t x_t' + \nu_{t-1})^{-1}(y_t - x_t' \theta_{t-1}) \]

The above equations can be derived using Bayes’ law. The derivations are standard so only some critical intermediate steps are presented here. The posterior density
\[ p(\theta, k|y_t) \propto p(y_t|\theta, k)p(\theta, k) \]

It can be derived from the right hand side that the posterior mean of the parameters is
\[ \theta_t \]
\[ = (x_t x_t' + \nu_{t-1})^{-1}(x_t y_t) \]
\[ = (x_t x_t' + \nu_{t-1})^{-1} x_t x_t' \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \]
\[ = \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \]

Note
\[ (y_t - x_t' \theta)'(y_t - x_t' \theta) + (\theta - \theta_{t-1})' \nu_{t-1}(\theta - \theta_{t-1}) \]
\[ = y_t' y_t - 2 \theta' y_t + \theta' x_t x_t' \theta + \theta' \nu_{t-1} \theta - 2 \theta' \nu_{t-1} \theta_{t-1} + \theta_t' \nu_{t-1} \theta_{t-1} \]
\[ = \theta'(x_t x_t' + \nu_{t-1}) \theta - 2 \theta'(x_t y_t + \nu_{t-1} \theta_{t-1}) + y_t' y_t + \theta_t' \nu_{t-1} \theta_{t-1} \]
\[ = (\theta - (x_t x_t' + \nu_{t-1})^{-1}(x_t y_t + \nu_{t-1}))(\theta - (x_t x_t' + \nu_{t-1})^{-1}(x_t y_t + \nu_{t-1}))' \]
\[ [y_t' y_t + \theta_t' \nu_{t-1} \theta_{t-1} - (\nu_{t-1} \theta_{t-1} + x_t y_t)'(x_t x_t' + \nu_{t-1})^{-1}(\nu_{t-1} \theta_{t-1} + x_t y_t)] \]

### C Deriving the Constant-Gain Learning Algorithm from Bayesian Updating\(^{32}\)

Agents perceive the following random walk model of coefficient variation
\[ \phi_t = \phi_{t-1} + \nu_t \quad E\nu_t \nu_t' = R_{11} \] \hspace{1cm} (53)
\[ y_t = \theta_t x_t + \xi_t \quad E\xi_t \xi_t' = R_{22} \] \hspace{1cm} (54)

\(^{32}\)The derivations follow Ljung (1991) and Sargent (1999) except that both of them use inconsistent notations.
Define $P_{t-1} = E[(\phi_{t-1} - \theta_{t-1})(\phi_{t-1} - \theta_{t-1})]$. The prior belief about $\theta_0$ are $N(\theta_0, P_{0|0})$. The posterior of $\theta_t$ can be represented by the basic Kalman filter, which takes the form of following recursions:\footnote{Note for the model considered here I have $\theta_{t|t-1} = \theta_{t-1|t-1}$, so I suppress the conditioned information set and use $\theta_t$ for both.}

\begin{align}
\theta_t &= \theta_{t-1} + L_t[y_t - x'_t\hat{\theta}_{t-1}] \\
L_t &= \frac{P_{t|t-1}x_t}{R_{2t} + x'_tP_{t|t-1}x_t} \\
P_{t+1|t} &= P_{t|t-1} - \frac{P_{t|t-1}x_t x'_t P_{t|t-1}}{R_{2t} + x'_t P_{t|t-1}x_t} + R_{1t+1}
\end{align}

Furthermore, agents are assumed to perceive $R_{1t} = \frac{g}{1-g}P_{t-1|t-1}$ and $R_{2t} = \frac{1}{g}$. Note $P_{t|t-1} = P_{t-1|t-1} + R_{1t} = \frac{1}{1-g}P_{t-1|t-1}$. Equations (56) – (57) become

\begin{align}
\theta_t &= \theta_{t-1} + L_t[y_t - x'_t\hat{\theta}_{t-1}] \\
L_t &= \frac{P_{t-1|t-1}x_t}{\frac{1-g}{g} + x'_t P_{t-1|t-1} x_t}
\end{align}

And equation (55) becomes

\begin{align}
P_{t+1|t} - R_{1t+1} &= P_{t|t-1} - \frac{P_{t|t-1}x_t x'_t P_{t|t-1}}{R_{2t} + x'_t P_{t|t-1}x_t} \\
P_{t|t} &= \frac{1}{1-g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x'_t P_{t-1|t-1}}{x'_t P_{t-1|t-1}x_t + \frac{1-g}{g}} \right]
\end{align}

The constant gain learning algorithm is following:\footnote{The learning rule using lagged data can be derived similarly as in Adam and Marcet (2010).}

\begin{align}
\theta_t &= \theta_{t-1} + gR^{-1}_t x_t(y_t - x'_t \hat{\theta}_{t-1}) \\
R_t &= R_{t-1} + g(x'_t x_t - R_{t-1})
\end{align}

Below I show the above two formulations are equivalent. Use $R_t^{-1} \equiv P_{t|t}$, equation (63) yields

\begin{align}
R_t^{-1} &= \left( (1-g)R_{t-1} + g x_t x'^{-1}_t \right) \\
&= \frac{1}{1-g} R^{-1}_{t-1} - \frac{1}{1-g} R^{-1}_{t-1} x_t \left[ x'_t \frac{1}{1-g} R^{-1}_{t-1} x_t + \frac{1}{g} \right]^{-1} x'_t \frac{1}{1-g} R^{-1}_{t-1} \\
&= \frac{1}{1-g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t x'_t P_{t-1|t-1}}{x'_t P_{t-1|t-1}x_t + \frac{1-g}{g}} \right]
\end{align}
From equation (64) to equation (65), the matrix inversion formula is used and stated in lemma 1 below. Specifically, it is applied with $A = (1 - g)R_{t-1}$, $B = x_t$, $C = g$, $D = x_t^*$.

Now I proceed to show the equivalence between equation (58) and (62). It suffices to show that $gR_{t-1}^{-1}x_t = L_t$.

$$gR_{t-1}^{-1}x_t \quad \begin{equation} (67) \end{equation}$$

$$= gP_{t|t}x_t \quad \begin{equation} (68) \end{equation}$$

$$= \frac{g}{1 - g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1}x_{t}x_{t}'P_{t-1|t-1}}{1 - \frac{g}{g} + x_{t}'P_{t-1|t-1}x_{t}} \right] x_t \quad \begin{equation} (69) \end{equation}$$

$$= \frac{1 - g}{g} + x_{t}'P_{t-1|t-1}x_{t} \quad \begin{equation} (70) \end{equation}$$

From equation (67) to (68), equation (61) is used.

**Lemma 1.** Let $A$, $B$, $C$ and $D$ be matrices of compatible dimensions, so that the product $BCD$ and the sum $A + BCD$ exist. Then

$$[A + BCD]^{-1} = A^{-1} - A^{-1}B[DA^{-1}B + C^{-1}]^{-1}DA^{-1} \quad \begin{equation} (71) \end{equation}$$

Proof: see Ljung and Soederstroem (1983) pp. 19. (Sketch: show the RHS of (71) multiplied by $A + BCD$ from the right is equal to identity matrix.)

### D  Proof of Proposition 4

Local stability of the Minimum State Variable (MSV) RE solution is determined by the stability of the following associated ordinary differential equations (ODEs)

$$\frac{d\zeta^m}{dt} = T_1(\zeta^m, \zeta^p) - \zeta^m$$

$$\frac{d\zeta^p}{dt} = T_2(\zeta^p, \zeta^p) - \zeta^p$$

where $T_1(\zeta^m, \zeta^p) = \frac{(\xi_1 \zeta^m - \xi_2 \zeta^m)(1 + \frac{\zeta^p}{\xi_3 R}) - \frac{1}{1 + \frac{\zeta^p}{\xi_3 R}} \xi_1 \zeta^m \zeta^p}{1 + \frac{\zeta^p}{\xi_3 R} - \xi_1 \zeta^m \frac{1}{\xi_3 + \frac{\zeta^p}{\xi_3 R}}}$, $T_2(\zeta^p, \zeta^p) = \frac{\xi_3 - \xi_2 \zeta^p}{1 - \frac{\zeta^p}{\xi_3 + \frac{\zeta^p}{\xi_3 R}}}$, $\xi_1 = \frac{1}{R} + \frac{1}{\eta} - \frac{1}{1 - \frac{1}{(1 - \tau) R}}$, $\xi_2 = (1 - \tau) \frac{1}{R} - \frac{1}{1 - \frac{1}{(1 - \tau) R}}$, $\xi_3 = \frac{1}{R} - \frac{1}{\eta}$.

Solving the fixed point of the T-map, I get $\bar{\zeta}^m = 0$, and $\bar{\zeta}^p = \frac{(1 - \frac{1}{R} \frac{1}{\eta})}{1 - \frac{1}{(1 - \tau) R}}$.

The E-stability condition requires that the eigenvalues of the Jacobian of the right hand side of the above ODEs are negative. Since $\zeta^m$ does not show up in the ODE for $\zeta^p$, the eigenvalues will be on the diagonal of the Jacobian matrix and only two partial derivatives, i.e., $\frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m}|_{\zeta^m = \bar{\zeta}^m, \zeta^p = \bar{\zeta}^p}$ and $\frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p}|_{\zeta^p = \bar{\zeta}^p}$, matter for the E-stability.

Fixing the arguments of the $T_2$-map at $\zeta^p$ yields
\[ T_2(\zeta^p, \zeta^p) = \frac{\xi_3 - \xi_2 \zeta^p}{1 - \frac{\xi_1 \zeta^p}{\xi_3 + \zeta^p}} \]
\[ = \frac{(\xi_3 + \frac{\zeta^p}{R})(\xi_3 - \xi_2 \zeta^p)}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \]
\[ = \frac{\zeta^p + (\frac{1}{R} - \xi_2)\xi_3^p - \xi_3^2(\zeta^p)^2}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \]

The derivative of \( T_2 \) with respect to \( \zeta^p \) evaluated at the RE value is

\[ \frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p} |_{\zeta^p=\zeta^p} = \frac{(\frac{1}{R} - \xi_2)\xi_3^p - \zeta^p \frac{2}{R} \xi_2}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \]
\[ - \left( \frac{1}{R} - \xi_1 \right) \left( \frac{\xi_3^2 + (\frac{1}{R} - \xi_2)\xi_3^p - \xi_3^2(\zeta^p)^2}{(\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p)^2} \right) \]
\[ = \frac{(\frac{1}{R} - \xi_2)\xi_3^p - \zeta^p \frac{2}{R} \xi_2}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} - \left( \frac{1}{R} - \xi_1 \right) \frac{\zeta^p}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \]
\[ = \frac{(\frac{1}{R} - \xi_2)\xi_3^p - \zeta^p \frac{2}{R} \xi_2 + \frac{1}{R} - \xi_1}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \]

(72)

Note \( \zeta^p \) is the fixed point of the \( T_2 \)-map, i.e., \( \zeta^p = \frac{\xi_3^2 + (\frac{1}{R} - \xi_2)\xi_3^p - \xi_3^2(\zeta^p)^2}{\xi_3 + (\frac{1}{R} - \xi_1) \zeta^p} \), is used when deriving the second equality above. Substituting for \( \xi_1, \xi_3 \), and \( \zeta^p \), I show that the denominator of the above derivative is positive.

\[ \frac{\xi_3 + \zeta^p}{R} (\frac{1}{R} - \xi_1) \]
\[ = \frac{1}{\eta} \frac{R - 1}{R} + \frac{\zeta^p}{R} \left( 1 - R \left( \frac{1}{R} + \frac{1}{\eta} \frac{R - 1}{R} \frac{1}{1 - \frac{1}{R}(1 - \tau)} \right) \right) \]
\[ = \frac{1}{\eta} \frac{R - 1}{R} - \left( \frac{1}{\eta} \frac{R - 1}{R} \right) \frac{1}{1 - \frac{1}{R}(1 - \tau)} \left( \frac{1}{\eta} \frac{R - 1}{R} \frac{1}{1 - \frac{1}{R}(1 - \tau)} \right) \]
\[ = \frac{1}{\eta} \frac{R - 1}{R} \left( \frac{R - \frac{1}{\eta}(1 - \tau)}{1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R})} \right) \]
\[ = \frac{1}{\eta} \frac{R - 1}{R} \left( \frac{R - \frac{1}{\eta}(1 - \tau)}{1 - \frac{1}{R} + \frac{1}{\eta}} \right) \]
\[ > 0 \]

(73)
I proceed to show that the derivative (72) is smaller than 1. Given (73), note
\[
\frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p} |_{\zeta^p = \bar{\zeta}^p} < 1
\]
is equivalent to
\[
(\frac{1}{R} - \xi_2)\xi_3 - \bar{\zeta}^p(\frac{2}{R} \xi_2 + \frac{1}{R} - \xi_1) < \xi_3 + (\frac{1}{R} - \xi_1)\bar{\zeta}^p
\]
Rearranging the above inequality yields
\[
(\frac{1}{R} - \xi_2 - 1)\xi_3 < 2\bar{\zeta}^p(\frac{\xi_2}{R} + \frac{1}{R} - \xi_1)
\]
Plugging \(\xi's\), it can be shown that the left hand side of the above inequality is negative because \(\xi_2 > 0\), \(\frac{1}{R} - 1 < 0\), and \(\xi_3 > 0\). It can be shown that the right hand side is exactly zero because \(\xi_1 = \frac{1}{R} + \xi_2\).

Now I turn to the first derivative. Recall \(T_1(\zeta^m, \zeta^p) = \frac{\xi_1\zeta^m - \xi_2\zeta^m(1 + \frac{\zeta^p}{R})}{1 + \frac{\zeta^p}{R} - \xi_1\zeta^p \frac{1}{\xi_3}}\). Taking the derivative of \(T_1\) with respect to \(\zeta^m\) and evaluate it at the RE belief yields
\[
\frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m} |_{\zeta^m = \bar{\zeta}^m, \zeta^p = \bar{\zeta}^p} = \frac{\xi_1 - \xi_2(1 + \frac{\zeta^p}{\xi_3 R})}{1 + \frac{\zeta^p}{\xi_3 R} - \xi_1\zeta^p \frac{1}{\xi_3}} = \frac{(\xi_1 - \xi_2)\xi_3 R - \xi_2\bar{\zeta}^p}{\xi_3 R + \zeta^p (1 - \xi_1 R)}
\]
I then show the above derivative is smaller than 1. Given (73), it is equivalent to show
\[
(\xi_1 - \xi_2)\xi_3 R - \xi_2\bar{\zeta}^p < \xi_3 R + \bar{\zeta}^p(1 - \xi_1 R)
\]
Rearranging the above inequality yields
\[
(\xi_1 - \xi_2 - 1)\xi_3 R < (\xi_2 + 1 - \xi_1 R)\bar{\zeta}^p
\]
Plugging in the parameters, it can be shown firstly the left hand side of the above inequality is negative because
\[
\xi_1 - \xi_2 - 1 = \frac{1}{R} + \frac{\xi_2}{R} - \xi_2 - 1 = \left(\frac{1}{R} - 1\right) (1 + \xi_2) < 0
\]
\(\xi_3 > 0\), and \(R > 0\). It can also be shown that the right hand side of (74) is zero because \(\xi_1 R = 1 + \xi_2\).
E Proof of Proposition 6

Define $s(\zeta^p_t) = \frac{\zeta_1 \zeta^p_t}{\zeta_3 + \zeta^p_t}$. Recall the $T_2$-map is

$$
T_2(\zeta^p_{t-1}, \zeta^p_t) = \frac{\xi_3 - \xi_2 \zeta^p_{t-1}}{1 - \xi_3 + \zeta^p_t} = \frac{\xi_3 - \xi_2 \zeta^p_{t-1}}{1 - s(\zeta^p_t)}
$$

$$
\simeq \zeta^p + \left( - \frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p} \right) \left( \zeta^p - \zeta^p_{t-1} \right) - \left( \frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p} \right) \left( \zeta^p - \zeta^p_t \right)
$$

where $-\frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p}$ and $\frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p}$ stand for $-\frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p}$ and $\frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p}$, respectively.

A sufficient condition to guarantee momentum in belief is

$$
-\frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p} \geq \frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p} \quad (75)
$$

Suppose $\zeta^p_{t-1} < \zeta^p_t \leq \tilde{\zeta}^p$, then $\tilde{\zeta}^p - \zeta^p_{t-1} > \tilde{\zeta}^p - \zeta^p_t \geq 0$. Given that the momentum condition (75) holds, we have

$$
\left( - \frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p} \right) \left( \zeta^p_{t-1} - \zeta^p \right) > \left( \frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p_{t-1}=\zeta^p, \zeta^p_{t-1}=\zeta^p} \right) \left( \zeta^p - \zeta^p_t \right)
$$

and hence $T_2(\zeta^p_{t-1}, \zeta^p_t) > \tilde{\zeta}^p$. Using the belief updating rule, agents will update their belief upward, i.e., $\zeta^p_{t+1} > \zeta^p_t$. Similarly, if $\zeta^p_{t-1} > \zeta^p_t \geq \tilde{\zeta}^p$, we have $T_2(\zeta^p_{t-1}, \zeta^p_t) < \tilde{\zeta}^p$ and $\zeta^p_{t+1} < \zeta^p_t$.

I proceed to calculate the two derivatives. We have

$$
-\frac{\partial T_2}{\partial \zeta^p_{t-1}} |_{\zeta^p} = \frac{\xi_2}{1 - s(\zeta^p_t)}
$$

and

$$
\frac{\partial T_2}{\partial \zeta^p_t} |_{\zeta^p} = \frac{\xi_3 - \xi_2 \tilde{\zeta}^p}{1 - s(\tilde{\zeta}^p)} \frac{s'(\tilde{\zeta}^p)}{1 - s(\tilde{\zeta}^p)} = \tilde{\zeta}^p \frac{s'(\tilde{\zeta}^p)}{1 - s(\tilde{\zeta}^p)}
$$

$\tilde{\zeta}^p$ is the fixed point of the $T_2$-map, i.e., $\frac{\xi_3 - \xi_2 \tilde{\zeta}^p}{1 - s(\tilde{\zeta}^p)} = \tilde{\zeta}^p$, is used in the last equality. So the condition (75) is equivalent to $\frac{-\xi_2}{1 - s(\zeta^p_t)} \geq \tilde{\zeta}^p \frac{s'(\tilde{\zeta}^p)}{1 - s(\tilde{\zeta}^p)}$ or

$$
\xi_2 \geq \tilde{\zeta}^p s'(\tilde{\zeta}^p) \quad (76)
$$

Calculating the derivative $s'(\tilde{\zeta}^p)$ as follows
\[ s'(\zeta) = \frac{\xi_1}{\xi_3 + \zeta} - \frac{\xi_1 \zeta}{\xi_3 + \zeta \frac{1}{R}} \frac{1}{R} \]

\[ = \frac{s(\zeta)}{\zeta} - \frac{s(\zeta^{\frac{1}{R}}) \frac{1}{R}}{\xi_3 + \zeta \frac{1}{R}} \]

So

\[ \zeta^p s'(\zeta) = s(\zeta^p) - \zeta^p s(\zeta^p) \frac{1}{R} \]

\[ = \frac{s(\zeta^p) \xi_3}{\xi_3 + \zeta^p \frac{1}{R}} \]

\[ = \frac{s(\zeta^p)(R - 1)}{(R - 1) + \eta \zeta^p} \quad (77) \]

The last equality comes from plugging in the expression for \( \xi_3 \). Below we use again that \( \zeta^p \) is the fixed point of the \( T_2 \)-map, i.e., \( \frac{\xi_3 - \xi_2 \zeta^p}{1 - s(\zeta^p)} = \zeta^p \), which implies that

\[ s(\zeta^p) = \xi_2 + 1 - \frac{\xi_3}{\zeta^p} \quad (78) \]

The inequality (76)
\[
\xi_2 \geq \zeta^p s'(\zeta^p)
\]

\[
\iff \xi_2 \geq \frac{s(\zeta^p)(R - 1)}{(R - 1) + \eta \zeta^p}
\]

(using (77))

\[
\iff \xi_2 \geq \left( \xi_2 + 1 - \frac{\xi_3}{\zeta^p} \right) \frac{(R - 1)}{(R - 1) + \eta \zeta^p}
\]

(using (78))

\[
\iff \xi_2 \geq \frac{(R - 1)}{(R - 1) + \eta \zeta^p} + \left( 1 - \frac{\xi_3}{\zeta^p} \right) \frac{(R - 1)}{(R - 1) + \eta \zeta^p}
\]

\[
\iff \frac{\eta \zeta^p}{(R - 1) + \eta \zeta^p} \geq \left( 1 - \frac{\xi_3}{\zeta^p} \right) \frac{(R - 1)}{(R - 1) + \eta \zeta^p}
\]

\[
\iff \xi_2 \eta \zeta^p \geq \left( 1 - \frac{\xi_3}{\zeta^p} \right) (R - 1)
\]

\[
\iff (1 - \tau) \frac{1}{\eta} \frac{R - 1}{R} \frac{1 - \frac{1}{R}(1 - \tau)}{1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R})}
\]

\[
\iff (1 - \tau) \frac{1}{\eta} \frac{R - 1}{R} \frac{1 - \frac{1}{R}(1 - \tau)}{1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R})} \geq \left( 1 - \frac{\frac{R - 1}{R}}{\frac{1/(1 - \tau)}{1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R})}} \right) (R - 1)
\]

\[
\iff \frac{(1 - \tau)}{R \eta + (1 - \tau)(R - \eta)} \geq \frac{(1 - \tau)(1 - \tau) - \frac{R - 1}{R} (1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R}))}{(1 - \frac{1}{R}(1 - \tau))}
\]

(79)

The right hand side of the above inequality is

\[
RHS = \frac{(1 - \frac{1}{R}(1 - \tau)) - \frac{R - 1}{R} \left( 1 + (1 - \tau)(\frac{1}{\eta} - \frac{1}{R}) \right)}{(1 - \frac{1}{R}(1 - \tau))}
\]

\[
= \frac{1 - \frac{1}{R} + \frac{1}{R} \tau - 1 + \frac{1}{R} - \frac{R - 1}{R}(1 - \tau)(\frac{1}{\eta} - \frac{1}{R})}{\frac{1}{R}(R - (1 - \tau))}
\]

\[
= \frac{1 - \frac{1}{R} + \frac{1}{R} \tau - 1 + \frac{1}{R} - \frac{R - 1}{R}(1 - \tau)(\frac{1}{\eta} - \frac{1}{R})}{\frac{1}{R}(R - (1 - \tau))}
\]

\[
= \frac{\eta R \tau - (R - 1)(1 - \tau)(R - \eta)}{\eta R (R - (1 - \tau))}
\]

(80)

So with equation (80) the inequality (79) becomes
\[
\frac{(1-\tau)}{R\eta + (1-\tau)(R-\eta)} \geq \frac{\eta R\tau - (R-1)(1-\tau)(R-\eta)}{\eta R(R-1-\tau))}
\]

\[
\iff \frac{(1-\tau)\eta R^2 - \eta R(R-1)(1-\tau)^2}{R\eta + (1-\tau)(R-\eta)} \geq \eta R\tau - (R-1)(1-\tau)(R-\eta)
\]

\[
\iff \frac{(1-\tau)\eta R^2 - \eta R(R-1)(1-\tau)^2}{R\eta + (1-\tau)(R-\eta)} \geq \eta R\tau - (R-1)(1-\tau)(R-\eta)
\]

\[
\iff (1-\tau)\eta R^2 - \eta R(R-1)(1-\tau)^2 \geq R^2\eta^2\tau + \eta R\tau(1-\tau)(R-\eta)
\]

\[
- R\eta (R-1)(1-\tau)(R-\eta) - (R-1)(1-\tau)^2(R-\eta)^2
\]

\[
\iff \frac{((R-\eta)^2(R-1) - \eta R + \eta R(R-\eta))(1-\tau)^2}{(\eta R^2 + R^2\eta^2 - \eta R^2 + \eta^2 R + (R-1)(R-\eta)\eta R}(1-\tau)
\]

\[
- R^2\eta^2
\]

\[
\geq 0 \quad (81)
\]

I proceed to simplify the inequality (81). Note the coefficient on \((1-\tau)^2\) in inequality (81) is

\[
(R-\eta)^2(R-1) - \eta R + \eta R(R-\eta) = (R^2 - 2\eta R + \eta^2) R - (R^2 - 2\eta R + \eta^2) - \eta R + \eta R^2 - \eta^2 R
\]

\[
= R^3 - \eta R^2 + \eta R - R^2 - \eta^2
\]

\[
= R^2(R-1) - \eta R(R-1) - \eta^2
\]

The coefficient on \((1-\tau)\) in inequality (81) is

\[
\eta R^2 + R^2\eta^2 - \eta R^2 + \eta^2 R + (R-1)(R-\eta)\eta R = R^2\eta(R-1) - \eta^2 R(R-1) + R^2\eta^2 + \eta^2 R
\]

\[
= R\eta(R-1) + 2\eta
\]

So the inequality (81) is equivalent to

\[
0 \leq \frac{(R^2(R-1) - \eta R(R-1) - \eta^2)(1-\tau)^2 + R\eta(R-1) + 2\eta)(1-\tau)}{R^2\eta^2}
\]

\[
\iff 0 \leq -\eta^2 \left((1-\tau)^2 - 2R(1-\tau) + R^2\right)
\]

\[
+ (R^2(R-1)(1-\tau) - R(R-1)(1-\tau)^2)\eta + R^2(R-1)(1-\tau)^2
\]

\[
\iff 0 \leq -\eta^2 (R - (1-\tau))^2 + R(R-1)(1-\tau)(R - (1-\tau))\eta + R^2(R-1)(1-\tau)^2
\]

\[
\iff (\eta(R - (1-\tau)) - \frac{R(R-1)(1-\tau)}{2})^2 \leq R^2(1-\tau)^2 \left(R(1-\tau) + \frac{(R-1)^2}{4}\right) \quad (82)
\]
Case 1: assuming \( \eta(R - (1 - \tau)) > \frac{R(R-1)(1-\tau)}{2} \) or equivalently

\[
\frac{2(R - (1 - \tau))}{R(R - 1)(1 - \tau)} > \frac{1}{\eta}
\]  

(83) implies

\[
\eta(R - (1 - \tau)) - \frac{R(R-1)(1-\tau)}{2} \leq R(1 - \tau)\sqrt{(R - 1) + \frac{(R - 1)^2}{4}}
\]

\[
\iff \frac{1}{\eta} \geq \frac{R(R-1)(1-\tau)}{2} + R(1 - \tau)\sqrt{(R - 1) + \frac{(R-1)^2}{4}}
\]

\[
\iff \frac{1}{\eta} \geq \frac{R - 1}{g(R)}
\]  

(84)

where \( g(R) = R\sqrt{(R - 1) + \frac{(R - 1)^2}{4}} + \frac{R(R-1)}{2} \)

Combining (83) and (84), I get

\[
\frac{2(R - (1 - \tau))}{R(R - 1)(1 - \tau)} > \frac{1}{\eta} \geq \frac{R - 1}{g(R)}
\]  

(85)

It can be shown that

\[
\frac{2(R - (1 - \tau))}{R(R - 1)(1 - \tau)} > \frac{R - 1}{g(R)}
\]

\[
\iff \frac{(R - 1)}{g(R)} > \frac{R(R-1)(1-\tau)}{2} + R(1 - \tau)\sqrt{(R - 1) + \frac{(R-1)^2}{4}}
\]

because the second inequality is true.

Case 2: assuming \( \eta(R - (1 - \tau)) \leq \frac{R(R-1)(1-\tau)}{2} \) or equivalently \( \frac{1}{\eta} \geq \frac{(R - (1 - \tau))}{R(R-1)(1-\tau)} \) (82) implies

\[
\frac{R(R - 1)(1 - \tau)}{2} - \eta(R - (1 - \tau)) \leq R(1 - \tau)\sqrt{(R - 1) + \frac{(R - 1)^2}{4}}
\]

Note \( \frac{R(R-1)(1-\tau)}{2} < R(1 - \tau)\sqrt{(R - 1) + \frac{(R-1)^2}{4}} \), so the above inequality holds. I get

\[
\frac{1}{\eta} \geq \frac{(R - (1 - \tau))}{R(R-1)(1-\tau)}
\]  

(86)

Combining (85) and (86), I get the momentum condition

\[
\frac{1-\tau}{R} \geq \frac{1}{g(R)}
\]

where \( g(R) = R\sqrt{(R - 1) + \frac{(R - 1)^2}{4}} + \frac{R-1}{2} \).
Robustness of the Qualitative Model Dynamics to an Alternative Specification of the Collateral Constraint

Some papers, say Boz and Mendoza (2013) use an alternative specification of collateral constraints that household borrowing is limited by a fraction of current collateral values rather than the expected liquidation value of collateral. This section shows that the qualitative dynamics, belief changes have a critical effect on collateral prices and the possibility of momentum dynamics, continues to hold in the model with this alternative specification of the collateral constraint.

Consider the following collateral constraint

\[ b_t^B \leq (1 - \tau) \frac{q_t}{R} H_t^B \]

The debt repayment is now \((1 - \tau) q_{t-1} H_{t-1}^B\) and still depends directly on agents’ price elasticity belief at \(t - 1\), i.e., \(\zeta_{t-1}^{p}\). Borrowers’ collateral demand becomes

\[ H_t^B = a + q_t - (1 - \tau) q_{t-1} H_{t-1}^B \]

Consider the deterministic case and without learning about the steady state. The collateral demand equation is

\[ H_t^B = a + q_t - \frac{(1 - \tau) q_{t-1}}{q_t - \frac{1}{R}(1 - \tau) q_t} H_{t-1}^B \]

The model equations with the alternative specification of the collateral constraint contain

\[ \hat{H}_t^B = \frac{1}{(1 - \frac{1 - \tau}{R})} \left[ \frac{1 - \tau}{R} \hat{q}_t - (1 - \tau) \hat{q}_{t-1} \right] + \hat{H}_{t-1}^B \quad (87) \]

\[ \hat{q}_t = \frac{1}{R} E_t^p \hat{q}_{t+1} + \frac{1}{\eta R} \hat{H}_t^B \]

Agents’ conditional expectations are \(E_t^p \hat{q}_{t+1} = \zeta_t^p \hat{H}_t^B\). Substituting the conditional expectations into the model equations, we get

\[ \hat{q}_t = \frac{1}{R} E_t^p \hat{q}_{t+1} + \frac{1}{\eta R} \hat{H}_t^B \]

\[ = \frac{1}{R} \zeta_t^p \hat{H}_t^B + \frac{1}{\eta R} \hat{H}_t^B \]

\[ = \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_t^B \]  

(89)
So we have
\[ \hat{q}_{t-1} = \left( \frac{1}{R} \zeta_{t-1}^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_{t-1} \] (90)

Combining equations (87), and (90), we get
\[ \hat{H}_t^B = \frac{1}{(1 - \frac{1 - \tau}{R})} \left[ \frac{1 - \tau}{R} \hat{q}_t - (1 - \tau)\hat{q}_{t-1} \right] + \hat{H}_{t-1} \]
\[ = \frac{1 - \frac{1 - \tau}{R}}{1 - \frac{1 - \tau}{R}} \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_t^B - \frac{1 - \frac{1 - \tau}{R}}{1 - \frac{1 - \tau}{R}} \left( \frac{1}{R} \zeta_{t-1}^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_{t-1} + \hat{H}_{t-1} \]

So
\[ \hat{H}_t^B = \frac{1 - \frac{1 - \tau}{R}}{1 - \frac{1 - \tau}{R}} \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_t^B \] (91)

Equations (89) and (91) yield the ALM for collateral prices under learning
\[ \hat{q}_t = \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \hat{H}_t^B \]
\[ = T_2 \left( \zeta_{t-1}^p, \zeta_t^p \right) \hat{H}_{t-1}^B \]

where
\[ T_2 \left( \zeta_{t-1}^p, \zeta_t^p \right) = \left( \frac{1}{R} \zeta_t^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \frac{1 - \frac{1 - \tau}{R}}{1 - \frac{1 - \tau}{R}} \left( \frac{1}{R} \zeta_{t-1}^p + \frac{1}{\eta} \frac{R - 1}{R} \right) \]

The actual price elasticity under learning, i.e., the \( T_2 \)-map, depends positively on current beliefs and negatively on past beliefs. So the qualitative learning dynamics under this alternative collateral constraint is similar to the one analyzed in the text.

G On the Bindingness of the Collateral Constraint

Similar to the Kiyotaki and Moore (1997) paper, two assumptions: (1) \( \beta^B < \frac{\beta^L}{\lambda} \)
(2) \( \frac{\tau}{1 - \frac{\tau}{R}} \left( \frac{(R-1)(1-\tau)}{aR} \right) > \frac{1}{\beta^H} - 1 \), have been made to ensure that the return to investment is higher than that to consumption and saving in a neighborhood of the steady state, so that the collateral constraint is always binding under RE. In the learning model agents evaluate the payoffs of different strategies using the subjective probability measure and there may be substantial deviations of beliefs and expectations. It is unclear that if the collateral constraint remains binding in this case without a formal check. This section confirms that the collateral constraint is indeed binding during the housing cycle over the year 2001-2008 in the learning model.
Consider a marginal unit of tradable consumption at date $t$ where $t$ runs from 2001Q1 to 2008Q4. The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding or save it and then invest. The payoff sequences for period $t, t+1, t+2, t+3, \ldots$ would be as follows.

\begin{align*}
\text{invest} & \quad 0, \quad \bar{e} \quad \frac{a + q_{t+1} - (1 - \tau)E_t^P q_{t+1}}{d_t} \quad \frac{\bar{e}}{d_{t+1}}, \\
& \quad \frac{a + q_{t+1} - (1 - \tau)E_t^P q_{t+1}}{d_t} \quad a + q_{t+2} - (1 - \tau)E_{t+1}^P q_{t+2} \quad \frac{\bar{e}}{d_{t+2}}, \ldots \\
\text{save} & \quad 0, \quad 0, \quad \frac{R_t}{d_{t+1}}, \\
& \quad \frac{a + q_{t+2} - (1 - \tau)E_{t+1}^P q_{t+2}}{d_{t+1}} \quad \frac{\bar{e}}{d_{t+2}}, \ldots \\
\text{consume} & \quad 1, \quad 0, \quad 0, \ldots
\end{align*}

(92)

In period $t$, the borrower invests 1 unit and can get $\frac{1}{d_t}$ unit of collateral with borrowing. In period $t+1$, he gets the production $\frac{a+q_{t+1}}{d_t}$. The non-tradable part $\frac{\bar{e}}{d_t}$ will be consumed and the tradable part $\frac{\bar{e}}{d_{t+1}}$ will be re-invested. There will be capital gain $\left(q_{t+1} - (1 - \tau)E_t^P q_{t+1}\right) \frac{1}{d_t}$. In period $t+1$, the sum of the tradable part and the capital gain, i.e., $\frac{a+q_{t+1} - (1 - \tau)E_t^P q_{t+1}}{d_t}$, will be reinvested, so $\frac{a+q_{t+1} - (1 - \tau)E_t^P q_{t+1}}{d_t} \frac{\bar{e}}{d_{t+1}}$ will be acquired in period $t+1$. In period $t+2$, $\frac{a+q_{t+1} - (1 - \tau)E_t^P q_{t+1}}{d_t} \frac{\bar{e}}{d_{t+1}}$ will be consumed; and so on.

In period $t$, if the borrower saves the 1 unit, then he will get $R_t$ in period $t+1$. He can acquire $\frac{R_t}{d_{t+1}}$ at the same period. In period $t+2$, he consumes $\frac{R_t}{d_{t+1}}$ and invests $\frac{R_t}{d_{t+1}} \frac{a+q_{t+2} - (1 - \tau)E_{t+1}^P q_{t+2}}{d_{t+1}}$; and so on.

Denote $P_t^I$, $P_t^S$ the payoff of the investment and saving strategy at period $t$, respectively. $Inv_t$ and $Sav_t$ stand for the discounted sum of period payoffs for the investment and saving strategy, respectively. So

\begin{align*}
Inv_t & = \sum_{i=t}^{\infty} \left(\beta B\right)^i P_t^I \\
Sav_t & = \sum_{i=t}^{\infty} \left(\beta B\right)^i P_t^S
\end{align*}

(95) (96)

The steady state value of $Inv_t$ is

\begin{align*}
Inv & = 0 + \beta B \frac{\bar{e}}{d} + \left(\beta B\right)^2 \frac{a + \tau q \bar{e}}{d} + \left(\beta B\right)^3 \left(\frac{a + \tau q}{d}\right)^2 \frac{\bar{e}}{d} + \ldots \\
& = \beta B \frac{\bar{e}}{d} \left(1 + \beta B + \left(\beta B\right)^2 + \ldots\right) \\
& = \beta B \frac{\bar{e}}{d} \frac{1}{1 - \beta B}
\end{align*}

48
and the steady state value of $Sav_t$ is

$$Sav = (\beta^B)^2 \frac{e}{d} + (\beta^B)^3 \frac{e}{d} + (\beta^B)^4 \frac{e}{d} + ...$$

$$= (\beta^B)^2 \frac{1}{1 - \beta^B}$$

To ensure a higher payoff for the investment strategy than that for saving at the steady state, i.e., $Inv > Sav$, we need $1 > \beta^B \frac{e}{d}$ or $\beta^L > \beta^B$. And

$$\beta^B \frac{1}{d} \frac{1}{1 - \beta^B} > 1 \quad (97)$$

is sufficient to ensure that the payoff for investment is higher than that for consuming it immediately. Substituting the steady state values, I get the inequality (97) is equivalent to

$$\frac{e}{1 - \frac{1}{R} (1 - \tau)} \frac{(R - 1) (1 - \tau)}{aR} > \frac{1}{\beta^B} - 1$$

Note $Inv_t$ and $Sav_t$ depend on the realization of future variables such as future prices. We are going to compare the conditional expectation of the payoffs of the three strategies at period $t$, $E_p^t Inv_t$ and $E_p^t Sav_t$. After log-linearization, $E_p^t \hat{Inv}_t$ and $E_p^t \hat{Sav}_t$ are identical to the log-linearization version of the discounted sum of the following payoff sequences,$^{35}$ respectively

**Invest**

$$0, \frac{e}{d_t}, \frac{a + \tau E_p^t q_{t+1}}{d_t}, \frac{e}{E_p^t d_{t+1}}, \frac{a + \tau E_p^t q_{t+1}}{d_t}, ..., \quad (98)$$

**Save**

$$0, \frac{q}{d_t}, \frac{R_t}{E_p^t d_{t+1}}, \frac{R_t a + \tau E_p^t q_{t+2}}{E_p^t d_{t+1}}, \frac{e}{E_p^t d_{t+2}}, ..., \quad (99)$$

where I replaced $q_{t+1} - (1 - \tau)E_p^t q_{t+1}, q_{t+2} - (1 - \tau)E_p^t q_{t+2}$... in (92) and (93) by $\tau E_p^t q_{t+1}, \tau E_p^t q_{t+2}$... in (98) and (99). I also replaced future down-payments $d_{t+1}, d_{t+2}, ...$ by $E_p^t d_{t+1}, E_p^t d_{t+2}, ...$.

Note at the steady state $\frac{q}{d} = \frac{1}{1 - \gamma}$. Log-linearizing the discounted sum of the payoff sequences (98) and (99) yields

$^{35}$Note this does not mean that the conditional expectation of the payoff sequence (92) and (93) are (98) and (99), respectively.
\[ I_{tv} \cdot E_t^{\bar{P}} \cdot \tilde{I}_{tv} = \beta B \frac{\bar{e}}{d} (-\hat{d}_t) + (\beta B)^2 \frac{\bar{e}}{d} \left( -\hat{d}_t + \frac{\tau}{1 - \frac{1}{R}} E_t^{\bar{P}} \hat{q}_{t+1} - E_t^{\bar{P}} \hat{d}_{t+1} \right) \]

\[ \quad + (\beta B)^3 \frac{\bar{e}}{d} \left( -\hat{d}_t + \frac{\tau}{1 - \frac{1}{R}} (E_t^{\bar{P}} \hat{q}_{t+1} + E_t^{\bar{P}} \hat{q}_{t+2}) - E_t^{\bar{P}} \hat{d}_{t+1} - E_t^{\bar{P}} \hat{d}_{t+2} \right) \]

\[ \quad + (\beta B)^4 \frac{\bar{e}}{d} \left( -\hat{d}_t - E_t^{\bar{P}} \hat{d}_{t+1} - E_t^{\bar{P}} \hat{d}_{t+2} - E_t^{\bar{P}} \hat{d}_{t+3} \right) \]

\[ \quad + \frac{\tau}{1 - \frac{1}{R}} (E_t^{\bar{P}} \hat{q}_{t+1} + E_t^{\bar{P}} \hat{q}_{t+2} + E_t^{\bar{P}} \hat{q}_{t+3}) \]

\[ \quad + \ldots \]

\[ = -\beta B \frac{\bar{e}}{d} \frac{1}{1 - \beta B} \hat{d}_t - \beta B \frac{\tau}{1 - \frac{1}{R}} E_t^{\bar{P}} \hat{q}_{t+1} + \sum_{i=1}^{\infty} (\beta B)^i E_t^{\bar{P}} \hat{q}_{t+i+1} \]

\[ = -\beta B \frac{\tau}{1 - \frac{1}{R}} \sum_{i=1}^{\infty} (\beta B)^i E_t^{\bar{P}} \hat{q}_{t+i+1} \]  \hspace{1cm} (100)

and

\[ S_{tv} \cdot E_t^{\bar{P}} \cdot \tilde{S}_{tv} = (\beta B)^2 R \frac{\bar{e}}{d} \left( \hat{R}_t - E_t^{\bar{P}} \hat{q}_{t+1} \right) \]

\[ + (\beta B)^3 R \frac{\bar{e}}{d} \left( \hat{R}_t - E_t^{\bar{P}} \hat{d}_{t+1} - E_t^{\bar{P}} \hat{d}_{t+2} + \frac{\tau q}{d} E_t^{\bar{P}} \hat{q}_{t+2} \right) \]

\[ + (\beta B)^4 R \frac{\bar{e}}{d} \left[ \hat{R}_t - E_t^{\bar{P}} \hat{d}_{t+1} - E_t^{\bar{P}} \hat{d}_{t+2} - E_t^{\bar{P}} \hat{d}_{t+3} \right] + \ldots \]

\[ = (\beta B)^2 R \frac{\bar{e}}{d} \frac{1}{1 - \beta B} \]

\[ \left( \hat{R}_t - \sum_{i=1}^{\infty} (\beta B)^{i-1} E_t^{\bar{P}} \hat{d}_{t+i} + \frac{\tau q}{d} \sum_{i=1}^{\infty} (\beta B)^i E_t^{\bar{P}} \hat{q}_{t+i+1} \right) \]  \hspace{1cm} (101)

Log-linearizing the down-payment yields

\[ \hat{d}_t = \frac{1}{1 - \frac{1}{R}} \left( \hat{q}_t - \frac{1 - \tau}{R} \left( E_t^{\bar{P}} \hat{q}_{t+1} - \hat{R}_t \right) \right) \]

and for \( i \geq 1 \)

\[ E_t^{\bar{P}} \hat{d}_{t+i} = \frac{1}{1 - \frac{1}{R}} \left( E_t^{\bar{P}} \hat{q}_{t+i} - \frac{1 - \tau}{R} \left( E_t^{\bar{P}} \hat{q}_{t+i+1} - E_t^{\bar{P}} \hat{R}_{t+i} \right) \right) \]

Define \( \chi_t \equiv (\chi_t^m, \chi_t^p)' \) where \( \chi_t^m \) and \( \chi_t^p \) are parameter estimates in the collateral holding equation. \( \beta'_t \) stands for agents’ belief for interest rates at period \( t \) and is
assumed to be updated via steady state learning for simplicity. \( x_t \) and \( \beta_t^r \) are updated recursively as follows

\[
\begin{align*}
x_t &= x_{t-1} + \gamma \left( S_t^H \right)^{-1} \left( \frac{1}{\hat{H}_{t-2}^B} \right) \left( \hat{H}_{t-1}^B - \left( 1 - \hat{H}_{t-2}^B \right) x_{t-1} \right) \\
S_t^H &= S_{t-1}^H + \gamma \left( \left( \frac{1}{\hat{H}_{t-2}^B} \right) \left( 1 - \hat{H}_{t-2}^B \right) - S_{t-1}^H \right) \\
\beta_t^r &= \beta_{t-1}^r + \gamma \left( \hat{R}_{t-1} - \beta_{t-1}^r \right)
\end{align*}
\]

where \( \gamma \) is the gain parameter.

We derive the collateral price forecast when \( i \geq 1 \)

\[
E_t^P \hat{q}_{t+i} = \phi_t^m + \phi_t^p E_t^P \hat{H}_{t+i}^B \\
= \phi_t^m + \phi_t^p \left( x_t^m \frac{1 - (x_t^p)^i}{1 - x_t^p} + (x_t^p)^{i-1} \hat{H}_t^B \right)
\]

Substituting the conditional expectations and the down-payments into (100) and (101), we get

\[
Inv \ast E_t^P \hat{Inv}_t = -\beta_t^B \frac{\tau}{d} \left[ \hat{d}_t - \beta_t^B \frac{\tau}{1 - \frac{\tau}{R}} E_t^P \hat{q}_{t+1} + \sum_{i=1}^{\infty} \left( \beta^B \right)^i E_t^P \hat{d}_{t+i} \right]
\]

\[
= -\beta_t^B \frac{\tau}{d} \left[ \hat{d}_t - \beta_t^B \frac{\tau}{1 - \frac{\tau}{R}} E_t^P \hat{q}_{t+1} \right]
\]

\[
+ \sum_{i=1}^{\infty} \left( \beta^B \right)^i \left( \frac{1}{1 - \frac{\tau}{R}} \left( E_t^P \hat{q}_{t+i} - \frac{1 - \tau}{R} \left( E_t^P \hat{q}_{t+i+1} - E_t^P \hat{R}_{t+i} \right) \right) \right)
\]

\[
= -\beta_t^B \frac{\tau}{d} \left[ \hat{d}_t - \beta_t^B \frac{\tau}{1 - \frac{\tau}{R}} E_t^P \hat{q}_{t+1} \right]
\]

\[
+ \sum_{i=1}^{\infty} \left( \beta^B \right)^i \left( \frac{1 - \frac{\tau}{R}}{1 - \frac{\tau}{R}} \left( E_t^P \hat{q}_{t+i} - \frac{1 - \tau}{R} \left( E_t^P \hat{q}_{t+i+1} - E_t^P \hat{R}_{t+i} \right) \right) \right)
\]

\[
= -\beta_t^B \frac{\tau}{d} \left[ \hat{d}_t - \beta_t^B \frac{\tau}{1 - \frac{\tau}{R}} E_t^P \hat{q}_{t+1} \right]
\]

\[
+ \sum_{i=1}^{\infty} \left( \beta^B \right)^i \left( \frac{1 - \frac{\tau}{R}}{1 - \frac{\tau}{R}} \left( E_t^P \hat{q}_{t+i} - \frac{1 - \tau}{R} \left( E_t^P \hat{q}_{t+i+1} - E_t^P \hat{R}_{t+i} \right) \right) \right)
\]

\[
\text{where the sum } \sum_{i=1}^{\infty} \left( \beta^B \right)^i E_t^P \hat{q}_{t+i+1} \text{ can be calculated as follows}
\]
\[
\sum_{i=1}^{\infty} (\beta^B)^i E_t^P \hat{q}_{t+i+1} = \sum_{i=1}^{\infty} (\beta^B)^i \left( \phi^m_i + \phi_t^p \left( x^m_{i-1} \frac{1 - x^p_{i-1}}{1 - x^p_i} + (x^p_i)^i \hat{H}_i^B \right) \right)
\]

\[
= \sum_{i=1}^{\infty} (\beta^B)^i \phi^m_i + \phi_t^p \sum_{i=1}^{\infty} (\beta^B)^i \left( \frac{x^m_i}{1 - x^p_i} \right) + \sum_{i=1}^{\infty} (\beta^B)^i \left( \hat{H}_i^B - \frac{x^m_i x^p_i \phi^p_t}{1 - x^p_i} \right)
\]

and the interest rate forecast
\[
E_t^P \hat{R}_{t+i} = \beta_t^r
\]

Comparing (100) with (101), we get
\[
Sav * E_t^P \tilde{S}av_t = \left( \beta^B \right)^2 \frac{R^E}{d} \frac{1}{1 - \beta^B} \left( \hat{R}_t - \sum_{i=1}^{\infty} (\beta^B)^i E_t^P \hat{d}_{t+i} + \frac{\tau_q}{d} \sum_{i=1}^{\infty} (\beta^B)^i E_t^P \hat{q}_{t+i+1} \right)
\]

\[
= \left( \beta^B \right)^2 \frac{R^E}{d} \frac{1}{1 - \beta^B} \hat{R}_t + R \left( Inv * E_t^P \tilde{Inv}_t + \beta^B \frac{1}{d} \frac{1}{1 - \beta^B} \hat{d}_t - (\beta^B)^2 \frac{\tau_q}{d} \frac{1}{1 - \beta^B} E_t^P \tilde{q}_{t+1} \right)
\]

Denote \( \Delta_t^1 \) and \( \Delta_t^2 \) the difference of the payoff between the investment strategy and the saving and the consumption strategy, respectively, so
\[
\Delta_t^1 = E_t^P \tilde{Inv}_t + \log (Inv) - \left( E_t^P \tilde{S}av_t + \log (Sav) \right)
\]

and
\[
\Delta_t^2 = E_t^P \tilde{Inv}_t + \log (Inv) - 1
\]

The parameterization in the quantitative section is used. In addition, we need to set a few more parameters. A different and smaller gain parameter \( \gamma = 0.03 \) is considered for updating \( \beta_t^r \) and \( \beta_t^r \) rather than 0.065 used in updating parameters in the collateral price equation. This is because \( \gamma = 0.065 \) will lead to \( x^p_t \) larger than 1, which we want to avoid. The discount factor \( \beta^B \) is set to 0.97. Numerical calculation shows that \( \Delta_t^1 \) and \( \Delta_t^2 \) are positive over the housing cycle in the learning model over the year 2001-2008.

\[36\] 

\[36\]For some alternative parameterizations, \( \Delta_t^1 \) and/or \( \Delta_t^2 \) can be negative for some periods. However, both \( \Delta_t^1 \) and \( \Delta_t^2 \) are positive for all \( t \) and for a large set of plausible and alternative parameterization, such as different gain parameter and/or different borrowers’ discount factor.