Comment on Assenza and Berardi “Learning in a Credit Economy” (2009, JEDC)
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Abstract

This comment shows that the “optimality” conditions in Assenza and Berardi (2009, JEDC) “Learning in a Credit Economy” imply that agents’ “optimal” choices are either suboptimal or infeasible. It presents the correct optimality conditions and discusses the effect on the E-stability condition of the REE. In addition, the different dynamics under the two sets of conditions is illustrated by considering an unexpected productivity impulse. Finally, under heterogeneous learning rules, numerical simulations illustrate that bankruptcy on the part of the borrowers arises sooner as they track the economy faster.

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1 Introduction

Assenza and Berardi (2009, JEDC, henceforth, AB) “Learning in a Credit Economy” replace the assumption of rational expectations in the basic version of the Kiyotaki and Moore (1997, JPE, henceforth KM) “Credit Cycles” model by assuming agents form expectations via adaptive learning. They find that under homogeneous learning among borrowers and lenders, the minimum state variable (MSV) solution for this economy is Expectationally Stable (E-stable) and hence can be learned by the agents. Their focus is that voluntary default of borrowers could occur due to the divergence of borrowers’ and lenders’ expectations when heterogeneous learning is allowed and uncertainty in terms of stochastic productivity is added.

Section 2 of this comment shows that the “optimality” conditions in AB imply that agents’ “optimal” choices are either suboptimal or infeasible and presents the correct optimality conditions. Section 3 discusses whether and how this may affect the Expectational-Stability (E-stability) condition of the REE. Furthermore, the different dynamics between the model with AB’s “optimality” conditions and with the correct ones is illustrated by considering responses to a temporary productivity impulse. Finally, section 4 presents numerical results on the timing of voluntary default of borrowers under heterogeneous learning rules.

2 Incorrect Optimality Conditions in AB and the Correct Ones

This section firstly derives the correct optimality conditions under learning and then shows that agents’ “optimal” choices in AB are either suboptimal or infeasible.

Reproducing a few major equations from AB. The flow-of-fund constraint of the farmer (or borrower) is

\[ q_t(K^F_t - K^F_{t-1}) + Rb_{t-1} + c^F_t = y^F_t + b_t \]  

(1)

where \( q_t, K^F_t - K^F_{t-1}, b_{t-1}, c^F_t \) and \( y^F_t \) are land (or collateral) price, the farmer’s investment in collateral holding, inherited debt holding, consumption and production, respectively. \( y^F_t = (a + \sigma)K^F_{t-1} \) is the production function of the farmer. \( aK^F_{t-1} \) is the tradable part of production and \( \sigma K^F_{t-1} \) the non-tradable part.

The farmer’s financing constraint is

\[ b_t \leq \frac{E_t q_{t+1}}{R} K^F_t \]

(2)

The maximum loan a farmer can get is the present value of expected liquidation value of collateral in next period.

Recall how the farmer makes his optimal decisions with respect to consumption, borrowing and land demand. Since return to investment in collateral holding is sufficiently high as shown in KM,¹

¹Recall the calculation in KM. Consider a marginal unit of tradable fruits at date \( t \). The borrower could consume it and get utility 1. Alternatively he could invest it in collateral holding and produce fruits. In the next period, he will consume the nontradable part of production and invest further the tradable part, and so forth. KM shows that the discounted sum of utilities of investing it at date \( t \) will exceed the utility of immediately consuming it, which is 1. Similarly, the return to investment will also be larger than the other choice, saving it for one period and then investing. Hence the collateral constraint will always be binding.
He prefers to borrow up to the maximum and invest in land. He will consume only the nontradable part of his product. His optimal consumption is

$$c_t^F = \varepsilon K_{t-1}^F$$  (3)

and optimal borrowing

$$b_t = \frac{E_t q_{t+1}}{R} K_t^F$$  (4)

Except for the initial period, every period the farmer’s inherited debt\(^2\) is

$$b_{t-1} = \frac{E_{t-1} q_t}{R} K_{t-1}^F$$  (5)

The debt repayment \(Rb_{t-1}\) is influenced by the expectation of collateral price at period \(t\) formed at period \(t-1\), i.e., \(E_{t-1} q_t\), in the case the borrowing constraint is binding.

Combining equations (1), (3), (4), (5) and the farmer’s production function, the farmer’s land demand equation can be derived as following

$$K_t^F = \frac{1}{\mu_t} (a + q_t - E_{t-1} q_t) K_{t-1}^F$$  (6)

where \(\mu_t\) is the down payment required to purchase a unit of collateral, which equals to \(q_t - E_t q_{t+1}/R\).

The gatherer’s (or lender’s) land demand equation is the same as that in AB and reproduced here

$$\mu_t = \frac{G'(K_t^G)}{R}$$  (7)

where \(G\) is the production function of the gatherer. Equation (7) states that the present value of marginal product of land is equal to user’s cost or the opportunity cost of lending to the farmer, which coincides with the downpayment.

The market clearing condition is

$$K_t^G + K_t^F = \overline{K}$$  (8)

Collateral prices and holding process under RE will be determined by equations (6), (7), and (8) given initial conditions. Under learning solving the collateral prices and collateral holding process requires additionally agents’ belief updating equations and initial beliefs.

AB has a different land demand equation of the farmer under homogenous learning, which is their equation (5) and reproduced here

$$K_t^F = \frac{a}{\mu_t} K_{t-1}^F$$  (9)

Comparing equation (9) with the correct equation (6), the former omits possible capital gains or losses in land holdings \((q_t - E_{t-1} q_t) K_{t-1}^F\). Under rational expectations equilibrium and deterministic environment, as in the original KM model, agents’ expectations about future land prices realize themselves due to perfect foresight. The value of land held from the previous period \(q_t K_{t-1}^F\) offsets

\(^2\)Is assumed to hold for the initial period and it is true if initially the economy is at the steady state and agents hold RE beliefs.
exactly the debt repayment $E_{t-1}q_t K_{t-1}^F$. The two collateral demand equations, i.e., equation (6) and equation (9), will coincide.

However, under adaptive learning, every period agents’ expectations about future collateral prices may not realize during the learning transition. The farmers may make persistent expectational errors, which produce capital gains or losses in their land holdings and generating additional variations in their net worth.

Where have the possible expectational errors due to adaptive learning gone? Implicit in AB is that the capital gains or losses are completely absorbed by consumption. This can be seen more clearly by deriving the implied consumption rule of the farmer in AB. Combining equation (1), (4), (5), (7), (9), the “optimal” consumption of the farmer implied by AB can be derived as following

$$c_t^F = c K_{t-1}^F + (q_t - E_{t-1}q_t) K_{t-1}^F$$

Note $q_t - E_{t-1}q_t K_{t-1}^F$ is non-zero as long as agents’ beliefs are not at the RE value.

The implications of equation (10) are following. When there is capital gain in the farmer’s collateral holding, i.e., $q_t K_{t-1}^F > E_{t-1}q_t K_{t-1}^F$, he will consume the capital gain instead of investing in collateral holding. He consumes part of tradable output beyond nontradable output, which is suboptimal relative to investing the capital gain in collateral holding, because the return to the latter is higher than consuming. When facing capital losses, i.e., $q_t K_{t-1}^F < E_{t-1}q_t K_{t-1}^F$, borrowers will consume only part of nontradable output. This implies further that they invest part of nontradable output in collateral holding, which is infeasible given that they are perishable and non-tradable.

3 Homogeneous Learning

This section redoes section 3 of AB and shows whether correcting the “optimality” conditions in AB may affect the E-stability condition of the REE under homogeneous learning among borrowers and lenders. Furthermore, the different dynamics under the AB “optimality” conditions and the correct ones is illustrated by considering an unexpected temporary productivity change.

3.1 The E-stability condition

Appendix A shows that linearizing equations (6), (7), and (8) yields

$$q_t = \gamma_0 + \gamma_1 K_{t-1}^F + \gamma_2 E_{t+1} q_t + \gamma_3 E_{t-1} q_t$$

$$K_t^F = \gamma_4 + \gamma_5 q_t + \gamma_6 E_{t+1} q_t$$

where $\gamma_0 = a$, $\gamma_1 = \frac{a^2 R^2}{2}$, $\gamma_2 = \frac{K a^2 R^2 + 1}{2 R}$, $\gamma_3 = -\frac{1}{2} [K a^2 R^2 - 1]$, $\gamma_4 = \tilde{K} - \frac{3}{(a R)^2}$, $\gamma_5 = \frac{2}{a R^2}$, and $\gamma_6 = \frac{4}{(a R)^3}$.

The model equations (11)-(12) differ from their counterpart, equations (9)-(10) in AB. The farmer’s land holding in the latter is a purely backward looking process, while in the former it also depends on one-period-ahead forecasts of land prices. Second, land prices here are influenced by expectations of land prices formed at two successive periods $t-1$ and $t$.

This model economy has the following Minimum State Variable (MSV) RE solution

$$q_t = \Phi_0 + \Phi_1 K_{t-1}^F$$

$$K_t^F = \Phi_2 + \Phi_3 K_{t-1}^F$$
where the parameters are to be determined. Following AB, agents’ perceived law of motion (PLM) is

\[ q_t = \phi_0 + \phi_1 K^F_t \]  \hspace{1cm} (15)

\[ K^F_t = \theta_0 + \theta_1 K^F_{t-1} \]  \hspace{1cm} (16)

Each period agents only need to form one-period-ahead forecasts of collateral prices. Their perceived law of motion for land is irrelevant for their decision. The parameters in agents’ law of motion for land will automatically converge to the RE value as long as the parameters in equation (15) converge to RE value.

Appendix B shows that the actual law of motion for land prices under learning is

\[ q_t = T_1(\phi_0, \phi_1) + T_2(\phi_0, \phi_1) K^F_{t-1} \]  \hspace{1cm} (17)

where \( T_1(\phi_0, \phi_1) = \frac{(\gamma_0 + \gamma_2 \phi_0 + \gamma_3 \phi_0) + \gamma_3 \phi_1}{1 - 2 \phi_0 \phi_1 + \phi_0^2} \) and \( T_2(\phi_0, \phi_1) = \frac{-\gamma_0 + \gamma_3 \phi_1}{1 - 2 \phi_0 \phi_1 + \phi_0^2} \).

Recall AB omit possible capital gains or losses in their land demand equation of the farmer. It is not obvious whether adding this term will affect the E-stability condition of the REE. Appendix B shows that the E-stability of the REE (11)-(12) requires

\[ \bar{K}a^2R^2 > \frac{2}{R} \]

Nevertheless, AB omitted KM’s assumptions for the gatherers’ production function \( G \), i.e., \( G' > 0, G'' < 0, G'(0) > aR > G'(\bar{K}) \) in the current context, which ensure that both farmers and gatherers are producing and holding (positive amount of) collateral at the steady state. In particular, assumption \( aR > G'(\bar{K}) \) implies \( \bar{K}a^2R^2 > 1 \) given that AB assume \( G(K^G_t) = 2(K^G_t)\frac{1}{2} \). Alternatively, this can be seen from the steady state value of collateral holdings of the farmer \( K^F = \bar{K} - (aR)^{-2} > 0 \). Finally, Appendix B also shows that \( \bar{K}a^2R^2 > 1 \) implies that the borrower’s collateral holding process at the MSV REE is stationary.

To sum up, the following proposition establishes the E-stability condition for the equilibrium (13) and (14).

**Proposition 1**

The MSV equilibrium for the economy represented by equations (11) and (12) is E-stable for any values of parameters satisfying that \( \bar{K}a^2R^2 > 1 \).

It appears that the condition is the “same” as that in AB. However, unlike in AB, the condition \( \bar{K}a^2R^2 > 1 \) here comes from the nonnegativity of steady state collateral holding of the farmer implied by KM, while the stationarity of the farmer’s collateral process is satisfied automatically.\(^5\) Capital gains (or losses) in AB are absorbed by consumption, so self-reinforcing deviations of prices and quantities from the REE in AB may be more harder than in my learning model for the same parameterization of the two models. Therefore, it is not surprising that the E-stability condition in my learning model is not looser than that in AB.

\(^3\)For linear RE models, it can be shown that both examples and counter-examples exist such that adding expectational errors changes the E-stability condition of the REE.

\(^4\)See equation (5) on p. 219 of KM

\(^5\)Kuang (2012) also provides interpretations to the E-stability condition for the MSV REE in a log-linearized version of the learning model.
3.2 Comparison of Model Dynamics

Despite the incorrectness of AB “optimality” conditions, an unexpected 1% temporary increase in the farmer’s productivity is considered to illustrate the different dynamics of the models under AB “optimality” conditions and the correct ones.

Assuming agents’ initial belief is at the RE value, the responses of prices and quantities in my learning model will be identical to those under RE. It can be shown that the initial increase of collateral price level is \( \frac{R(K\alpha^2R^2-1)}{2(R-1)} \) multiple of the percentage change of the farmer’s productivity with correct optimality conditions and \( \frac{2(K\alpha^2R^2-1)}{\alpha^2R^2(K\alpha^2R^2+1)} \) with AB “optimality” conditions. Furthermore, the former is larger than the later under all admissible parameterizations, so the impact response of prices under AB is dampened.

Agents’ perceived law of motion is \( q_t = \phi_t x_t + \eta_t \) with \( \phi_t \equiv (\phi_{0,t} \phi_{1,t})' \), where \( \eta_t \) is the regression residual. In both AB and my learning model, they are assumed to use a constant gain RLS learning algorithm

\[
\begin{align*}
\phi_t &= \phi_{t-1} + gR_t^{-1}x_{t-1}(q_{t-1} - \phi_{t-1}x_{t-1}) \\
R_t &= R_{t-1} + g(x_{t-1}x_{t-1}' - R_{t-1})
\end{align*}
\]

to learn parameters in their PLM, i.e., equations (15)-(16), where the state variables \( x_t = (1 K_{t-1}' \)'\). With such learning rule, agents discount old data and give relative larger weight to recent data, keeping track of possible structural changes of the economy.

Except for the impact period, in real time the actual law of motion for collateral prices in my learning model is

\[
\begin{align*}
q_t &= T_1(\phi_{0,t-1}, \phi_{0,t}, \phi_{1,t}) + T_2(\phi_{1,t-1}, \phi_{1,t})K_t^F \\
K_t^F &= \frac{\gamma_4 + \gamma_6q_{0,t}}{1 - \gamma_6\phi_{1,t}} + \frac{\gamma_5}{1 - \gamma_6\phi_{1,t}}q_t
\end{align*}
\]

where \( T_1(\phi_{0,t}, \phi_{1,t}) = \frac{\gamma_6q_{0,t}}{1 - \gamma_2\phi_{1,t} \gamma_5q_{0,t-1}} + \gamma_2\phi_{1,t-1} \) and \( T_2(\phi_{0,t}, \phi_{1,t}) = \frac{\gamma_1 + \gamma_3\phi_{1,t-1}}{1 - \gamma_2\phi_{1,t} \gamma_5q_{0,t-1}} \).

With AB “optimality” conditions, the actual law of motion for collateral prices is

\[
\begin{align*}
q_t &= (\zeta_0 + \zeta_2\phi_{0,t} + \zeta_2\phi_{1,t}) + (\zeta_1 + \zeta_2\phi_{1,t})K_t^F \\
K_t^F &= \zeta_3 + \zeta_4K_{t-1}^F
\end{align*}
\]

Here \( \zeta 's \) denote \( \gamma 's \) in AB, for example, \( \zeta_0 \) here represents \( \gamma_0 \) in the AB paper.

Parameterization

The same parameterization is used for AB and my learning model. The productivity of the farmer is normalized at one, i.e., \( a = 1 \). Following KM, the parameter \( K \) is set such that farmers hold two third of the collateral in the steady state. The gross interest rate is set to 1.06. The standard deviation of the productivity shock is set to 0.00712. The functional form of the production function chosen by AB implies that the collateral intensity equals to 0.5 in the gatherer’s production function. For the learning algorithm, the gain parameter is set to 0.02.\(^6\) This can be interpreted as that agents use data of 50 quarters in their estimate. Agents’ initial belief is set to the RE value.

\(^6\)Impulse response functions under the learning model with different gain parameters are available upon request.
**Impulse Response Functions**

Figure 1 displays the impulse response functions of collateral prices, the farmer’s land demand, forecasts of collateral prices and lending for 10 years under rational expectations, my learning model, and AB to 1% unexpected temporary increase in the farmer’s productivity.

Comparing first the responses under my learning model with those under RE. Since agents’ initial belief is at the RE level, they have correct forecast functions for collateral prices initially. In the impact period, responses of all variables in my learning economy are identical to those under RE. Interestingly, the response of my learning model is amplified and propagated further relative to the rational expectations version of the model due to belief revisions of the agents and partial validation of agents’ beliefs by subsequently realized prices. After the impact period, a positive surprise in collateral prices induce agents to revise their belief upward and they become more optimistic about future prices than under RE. Based on more optimistic expectations about future collateral prices, the credit limit will be relaxed and larger loans will be granted by lenders. With a larger borrowing capacity, borrowers can afford more and increase their demand on collateral, which boosts collateral prices further up. The realized collateral price reinforces agents’ initial optimism and leads to further upward adjustment or optimism. The dynamic interaction of agents’ beliefs and price realizations generates prolonged periods of expansions of prices and quantities.

Further increase in collateral prices will be choked off due to endogenous model dynamics. Equation (6) implies that borrowers’ will start to reduce collateral holdings when the capital gain of collateral holdings falls short of the increase of the downpayment (relative to its steady state value $a$), i.e., when $q_t - E_{t-1}q_t < a$. Collateral prices will decline subsequently. Agents will revise their beliefs downward and they become pessimistic about future prices. Based on the pessimism, credit limits are tightened by the lenders, which reduces further borrowers’ demand on collateral. The realized prices reinforce agents’ pessimism and leads to further downward adjustment of beliefs. Collateral prices and quantities cycle around the steady state for many periods and then converge to the steady state. This is because the learning model with collateral constraint has an intrinsic property that it may generate momentum or strong persistence in belief changes and hence in price changes.\(^7\)

Now comparing my learning model with the learning model with AB “optimality” conditions. In the impact period, the response of prices and quantities under AB are much dampened.\(^8\) The dampened response in AB is due to not only smaller initial response but also its weaker ability to reinforce agents’ beliefs in subsequent periods when borrowers consume the capital gains instead of investing in land holdings. Also note during the price decline phase, the prices in AB decrease less promptly because the capital losses are absorbed by reducing consumption so the farmers’ net worth and hence collateral demand fall by less.\(^9\)

\(^7\)More detailed analysis of the transitional dynamics of a log-linearized version of the learning model is provided in Kuang (2012).

\(^8\)The dampened response in AB learning model is robust for all admissible parameterizations and for different gain parameters.

\(^9\)Note in the basic version of the KM model, the steady state leverage ratio (or debt/asset ratio) equals to $1/R$ and is very large for $R$ close to 1. A small change in collateral prices will generate large fluctuations of collateral demand. Nevertheless, this figure can be used to illustrate the different dynamics between the model with AB “optimality” conditions and with correct optimality conditions.
4 Heterogeneous Learning and Voluntary Default

Beyond the case with homogeneous learning, AB considered that the farmers and gatherers learn independently from each other, and hence possibly have heterogeneous expectations about future collateral prices. In particular, they considered borrowers and lenders have different gain parameters in their learning rules and hence different speed of learning. They show that borrowers will choose to default voluntarily when their expectation about future collateral price falls below the expectation of lenders. It can be shown that correcting the borrower’s land demand equation does not affect their voluntary bankruptcy decision and the bankruptcy condition. Under heterogeneous expectations, the correct borrowers’ land demand equation is

\[ K_t^F = \frac{(a + q_t - \hat{q}_{t-1}^eG_t)}{\mu_{t,eG_t}^e} K_{t-1}^F \]

instead of equation (19) in AB. The lenders’ land demand equation remains equation (20) of AB. Under heterogeneous learning, it can be shown that the E-stability condition for the MSV equilibrium is still the same as that under homogeneous learning.

Below numerical simulations are provided to illustrate how the timing of voluntary default decisions by borrowers depends on the gain parameters of the agents and the size of the productivity shock under heterogeneous learning rules. In the last paragraph of their page 1167, AB claimed that “The bigger is the difference in the gain parameters and the greater the variance of the productivity shocks, the sooner bankruptcy arises.” (henceforth AB’s claim (Δ)). The simulation results here suggest that their claim is not true in the learning model with correct optimality conditions.

The simulation is performed as following. At period 1, borrowers and lenders are assumed to start with identical and RE beliefs. The learning rules of the borrowers and lenders of AB are adopted here. The gain parameter in the lenders’ learning rule is set to 0.01. The economy is assumed at the steady state initially. The average defaulting time of the borrowers is calculated under combinations of different gain parameters of borrowers ranging from 0.001 to 0.038 and different standard deviation of productivity shock from 1 multiple to 1/10 of 0.00712 based on 5000 repetitions. In each repetition the voluntary defaulting time is recorded as the first period when the expectation of borrowers is smaller than that of lenders. The seed of the random number generator is set to 21. The same 5000 series of shocks are used for all parameter combinations.

Note in two scenarios borrowers will choose to default immediately, i.e., in period 2, in this model. The first scenario is a positive initial shock combined with a smaller gain parameter of borrowers. This leads to that borrowers update their beliefs upward by less than lenders and

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10 For further research, an alternative approach to model default in the model with learning and collateral constraint is to assume homogeneous expectations among borrowers and lenders but multi-period debt contract. The outstanding debt is determined by lenders’ expectations and subjective evaluation of the collateral values at the initiating date. Due to learning, agents’ price forecast functions and subjective valuation of collateral may change over the life of the contract and hence may have incentive to default.

11 Alternatively, one may start with different initial beliefs than RE beliefs or start with higher borrowers’ expectations than lenders’.

12 In the first paragraph of page 1164 in AB, they mentioned that the farmer pays a cost \( C \) for a preliminary contract to buy a house such as in Italy but assume \( C = 0 \) in their paper. In a more realistic setting, all or at least most immediate defaults may be eliminated if the farmer need to pay a sufficiently large \( C > 0 \), because then the farmer will decide to default voluntarily when their expectation about future land price falls by more than certain positive amount below the expectation of the lender. One may conjecture that the average defaulting time calculated below is an increasing function of \( C \) since the bankruptcy condition becomes tighter. The comment follows AB and does not pursue the case with \( C > 0 \).
hence have lower price expectations in the second period. The second is a negative initial shock combined with a larger gain parameter of borrowers. In this case, borrowers will update their beliefs downward by more than those by lenders, so borrowers’ expectations will fall below lenders’ in the second period.

Table 1 reports the average defaulting time without excluding the cases of immediate voluntary default, which accounts for about half of the repetitions, and excluding them in table 2. For both tables, Panel A reports the scenarios in which borrowers have a smaller gain parameter than lenders and Panel B borrowers have a larger gain parameter.

The simulation results suggest that AB’s claim (\(\Delta\)) is not true in the learning model with correct optimality conditions. For example, comparing (1,1) element with (2,2) element of Panel A of table 1. The former (latter) reports the average defaulting time when both the variance of productivity shock and the gain difference between borrowers and lenders are larger (smaller), i.e., the standard deviation is 0.00712 (0.9*0.00712) and the gain difference is 0.009 (0.008). The average defaulting time under the former is 10.755, while that under the latter is 10.719. This contradicts with AB’s claim (\(\Delta\)). More such examples can be found in both panels of both tables.

Further results are summarized as following. First, the average voluntary defaulting time depends negatively on the gain parameter of borrowers instead of the gain difference of borrowers and lenders holding other parameters constant. This can be seen from each row of both panels in table 1, which reports the average defaulting period when the gain parameter of the borrowers is varied holding other parameters constant. This result is robust for different size of productivity shock, or different gain parameters of lenders.\(^{13}\) Bankruptcy arises sooner as the borrowers track the economy faster and update their belief more aggressively. This also holds after removing immediate defaults as in table 2. Not surprisingly, removing the immediate defaults will increase the average defaulting time for all parameterizations.

Second, the timing of default depends non-monotonically on the variance of the productivity shock holding other parameters constant. This can be seen from each column of both tables when the size of productivity shock varies holding other parameters constant.

5 Conclusions

This comment shows that the “optimality” conditions in Assenza and Berardi (2009, JEDC) “Learning in a credit economy” imply that agents’ “optimal” choices are either suboptimal or infeasible. It presents the correct optimality conditions and discusses how this affects the E-stability condition of the REE. In addition, the responses of prices and quantities to an unexpected temporary impulse in the farmers’ productivity are much dampened in AB relative to those in the model with correct optimality conditions.

Under heterogeneous learning, AB claimed that “The bigger is the difference in the gain parameters and the greater the variance of the productivity shocks, the sooner bankruptcy arises.” Numerical simulations are provided to show that their claim is not true in the learning model with correct optimality conditions. Furthermore, the results suggest that bankruptcy on the part of the borrowers arises sooner as they track the economy faster and the timing of bankruptcy depends non-monotonically on the variance of the productivity shock.

\(^{13}\)The results for different gain parameters of lenders/borrowers or for finer grids of the gain parameter of the borrowers are available upon request.
References


Appendix

A Linearizing the learning economy with correct optimality conditions

Reproducing equations (6) and (7) here

\[ K_t^F = \frac{a + q_t - E_{t-1} q_t}{\mu_t} K_{t-1}^F \]  \hspace{1cm} (24)

\[ K_t^G = G^{-1}(R_{t}) \tan \]  \hspace{1cm} (25)

where \( \mu_t = q_t - \frac{1}{R_t}E_t q_{t+1} \). The aggregate supply of land is fixed, i.e., \( K_t^G + K_t^F = \bar{K} \). Assuming that the gatherer’s production function is Cobb-Douglas, i.e., \( G(K_t^G) = 2 \sqrt{K_t^G} \), equation (25) leads to \( K_t^G = (R_{t})^{-2} \).

The steady state of the system is \( a = \mu, q = \frac{aR}{R-1}, K_t^G = (aR)^{-2} \) and \( K_t^F = \bar{K} - (aR)^{-2} \). I proceed to linearize the system around its steady state. Rearranging equation (24),

\[ K_{t-1}^F = \frac{K_t^F \mu_t}{a + q_t - E_{t-1} q_t} \]

\[ = \frac{(\bar{K} - \frac{1}{R_{t}})^{\mu_t}}{a + q_t - E_{t-1} q_t} \]

\[ = \frac{\bar{K}(q_t - \frac{1}{R_{t}}E_t q_{t+1})^2 - \frac{1}{R_{t}^2}}{(a + q_t - E_{t-1} q_t)(q_t - \frac{1}{R_{t}}E_t q_{t+1})} \]  \hspace{1cm} (26)

Linearizing the RHS of equation (26),

\[ K_{t-1}^F \approx \bar{K} - \frac{1}{a^2 R^2} + \frac{2\bar{K}}{a} + \left(\frac{\bar{K}}{a} + \frac{1}{a^3 R^2}\right)^2 + \left(\frac{\bar{K}}{a} + \frac{1}{a^3 R^2}\right) \left(q_t - \frac{aR}{R-1}\right) \]

\[ + \left[\frac{2\bar{K}}{aR} + \left(\frac{\bar{K}}{aR} - \frac{1}{(aR)^3}\right)^2 \left(E_t q_{t+1} - \frac{aR}{R-1}\right) + \left(\frac{\bar{K}}{a} - \frac{1}{a^3 R^2}\right) \left(E_{t-1} q_t - \frac{aR}{R-1}\right) \right] \]

\[ = \bar{K} - \frac{1}{(aR)^2} + \frac{2}{a^3 R^2} \left(q_t - \frac{aR}{R-1}\right) - \frac{1}{aR} \left[\frac{\bar{K}}{a} + \frac{1}{(aR)^2}\right] \left(E_t q_{t+1} - \frac{aR}{R-1}\right) \]

\[ + \frac{1}{a} \left[\bar{K} - \frac{1}{a^2 R^2}\right] \left(E_{t-1} q_t - \frac{aR}{R-1}\right) \]

Rearranging the above equation yields

\[ q_t = \gamma_0 + \gamma_1 K_{t-1}^F + \gamma_2 E_t q_{t+1} + \gamma_3 E_{t-1} q_t \]  \hspace{1cm} (27)

where \( \gamma_0 = a, \gamma_1 = \frac{a^3 R^2}{2}, \gamma_2 = \frac{\bar{K} a^2 R^2 + 1}{2R}, \) and \( \gamma_3 = -\frac{1}{2} \left[\bar{K} a^2 R^2 - 1\right] \).
Combining $K^G_t = \tilde{K} - K^F_t$ and equation (25) leads to

\[ K^F_t = \tilde{K} - (Rq_t - E_tq_{t+1})^{-2} \]
\[ \approx \tilde{K} - \frac{1}{(aR^2)} + 2\frac{R}{(aR^3)}(q_t - \frac{aR}{R-1}) - \frac{2}{(aR^3)}(E_tq_{t+1} - \frac{aR}{R-1}) \]
\[ = \gamma_4 + \gamma_5q_t + \gamma_6E_tq_{t+1} \]

where $\gamma_4 = \tilde{K} - \frac{3}{(aR^2)}$, $\gamma_5 = \frac{2}{aR^3}$, and $\gamma_6 = -\frac{2}{aR^3}$. Linearized equations (27) and (28) govern the dynamics of the farmer’s land holding and land price for the learning economy.

**B Derivation of the Minimum State Variable (MSV) RE Solution and Proposition 1**

Reproducing the linearized system of equations (11)-(12) for the learning economy here

\[ q_t = \gamma_0 + \gamma_1K^F_{t-1} + \gamma_2E_tq_{t+1} + \gamma_3E_{t-1}q_t \]
\[ K^F_t = \gamma_4 + \gamma_5q_t + \gamma_6E_tq_{t+1} \]

where $\gamma_0 = a$, $\gamma_1 = \frac{a^2R^2}{2}$, $\gamma_2 = \frac{\bar{K}a^2R^2 + 1}{2R}$, $\gamma_3 = -\frac{1}{2}[\bar{K}a^2R^2 - 1]$, $\gamma_4 = \tilde{K} - \frac{3}{(aR^2)}$, $\gamma_5 = \frac{2}{aR^3}$, and $\gamma_6 = -\frac{2}{aR^3}$.

It has the following MSV RE solution

\[ q_t = \Psi_0 + \Psi_1K^F_{t-1} \]
\[ K^F_t = \Psi_2 + \Psi_3K^F_{t-1} \]

where $\Psi_0$, $\Psi_1$, $\Psi_2$ and $\Psi_3$ are to be determined. The perceived law of motion of agents is

\[ q_t = \phi_0 + \phi_1K^F_{t-1} \]
\[ K^F_t = \theta_0 + \theta_1K^F_{t-1} \]

Agents’ conditional expectations are $E_tq_{t+1} = \phi_0 + \phi_1K^F_{t}$, and $E_{t-1}q_t = \phi_0 + \phi_1K^F_{t-1}$.

The agents only need to form expectations of collateral prices. Agents’ PLM for land do not matter for their decisions. So the parameters in the actual law of motion for land will automatically converge to the RE value as long as the parameters in equation (31) converge to the RE level.

Substituting the conditional expectations into equations (29)-(30), the actual law of motion for the farmer’s land demand can be derived as following

\[ K^F_t = \gamma_4 + \gamma_5q_t + \gamma_6E_tq_{t+1} \]
\[ = \gamma_4 + \gamma_5q_t + \gamma_6(\phi_0 + \phi_1K^F_{t}) \]
\[ = \frac{\gamma_4 + \gamma_6\phi_0}{1 - \gamma_6\phi_1} + \frac{\gamma_5}{1 - \gamma_6\phi_1}q_t \]

\[ (33) \]
and the actual law of motion for collateral price

\[
q_t = \gamma_0 + \gamma_1 K_{t-1}^F + \gamma_2 E_t q_{t+1} + \gamma_3 E_{t-1} q_t
\]

\[
= \gamma_0 + \gamma_1 K_{t-1}^F + \gamma_2 (\phi_0 + \phi_1 K_t^F) + \gamma_3 (\phi_0 + \phi_1 K_{t-1}^F) + \gamma_4 (\phi_0 + \phi_1 K_t^F)
\]

\[
= \gamma_0 + \gamma_2 \phi_0 + \gamma_3 \phi_0 + (\gamma_1 + \gamma_3 \phi_1) K_{t-1}^F + \gamma_2 \phi_1 K_t^F
\]

\[
+ \gamma_2 \phi_1 \left( \frac{T_4 + \gamma_6 \phi_0}{1 - \gamma_6 \phi_1} + \frac{\gamma_5}{1 - \gamma_6 \phi_1} q_t \right)
\]

\[
= T_1(\phi_0, \phi_1) + T_2(\phi_0, \phi_1) K_t^F
\]

where \( T_1(\phi_0, \phi_1) = \frac{(\gamma_0 + \gamma_2 \phi_0 + \gamma_3 \phi_0) + \gamma_4 \phi_1 K_{t-1}^F \gamma_6 + \gamma_6 \phi_0}{1 - \gamma_2 \phi_1 - \gamma_6 \phi_0} \) and \( T_2(\phi_0, \phi_1) = \frac{\gamma_1 + \gamma_3 \phi_1}{1 - \gamma_2 \phi_1 - \gamma_6 \phi_0} \).

Combining (33) and (34) yields

\[
K_t^F = \frac{\gamma_4 + \gamma_6 \phi_0}{1 - \gamma_6 \phi_1} + \frac{\gamma_5}{1 - \gamma_6 \phi_1} q_t
\]

\[
= \gamma_4 + \gamma_6 \phi_0 \left( \frac{1}{1 - \gamma_6 \phi_1} + \frac{\gamma_5}{1 - \gamma_6 \phi_1} \right) T_1(\phi_0, \phi_1) + T_2(\phi_0, \phi_1) K_{t-1}^F
\]

\[
= \gamma_4 + \gamma_6 \phi_0 \left( \frac{1}{1 - \gamma_6 \phi_1} + \frac{\gamma_5}{1 - \gamma_6 \phi_1} T_1(\phi_0, \phi_1) \right) + \gamma_5 T_2(\phi_0, \phi_1) K_{t-1}^F
\]

(35)

At the fixed point, we have \( \phi_1 = T_2 = \Psi_1 \), which yields

\[
\Psi_1 = \frac{\gamma_1}{1 + \gamma_1 \gamma_6 - \gamma_3}
\]

\[
= \frac{\gamma_1}{\frac{K a^2 R^2 + 1}{2} - 1}
\]

Note \( \phi_0 \) does not appear in \( T_2 \). So only two partial derivatives, \( \frac{\partial T_2}{\partial \phi_0} \big|_{\phi_0=\psi_0, \phi_1=\psi_1} \) and \( \frac{\partial T_1}{\partial \phi_0} \big|_{\phi_0=\psi_0, \phi_1=\psi_1} \), matter for the E-stability conditions.

\[
T_2(\phi_0, \phi_1) = \frac{(\gamma_1 + \gamma_3 \phi_1)(1 - \gamma_6 \phi_1)}{1 - \gamma_6 \phi_1 - \gamma_2 \gamma_5 \phi_1}
\]

\[
= \frac{\gamma_1 + \gamma_3 \phi_1}{1 - \gamma_6 \phi_1 - \gamma_2 \gamma_5 \phi_1}
\]

The partial derivative \( \frac{\partial T_2}{\partial \phi_1} \big|_{\phi_0=\psi_0, \phi_1=\psi_1} \) is

\[
\frac{\partial T_2}{\partial \phi_1} \big|_{\phi_0=\psi_0, \phi_1=\psi_1} = \frac{\gamma_3 - \gamma_1 \gamma_6 - 2 \gamma_3 \gamma_6 \phi_1}{1 - \gamma_6 \phi_1 - \gamma_2 \gamma_5 \phi_1}
\]

\[
+ \frac{\gamma_3 - \gamma_1 \gamma_6 - 2 \gamma_3 \gamma_6 \phi_1 + \phi_1 (\gamma_6 + \gamma_2 \gamma_5) |_{\phi_0=\psi_0, \phi_1=\psi_1}}{1 - \gamma_6 \phi_1 - \gamma_2 \gamma_5 \phi_1}
\]

\[
= \frac{\gamma_3 - \gamma_1 \gamma_6 - \gamma_3 \gamma_6 \Psi_1}{1 - \gamma_6 \Psi_1 - \gamma_2 \gamma_5 \Psi_1}
\]
The E-stability for the MSV RE equilibrium requires that \( \frac{\partial T_2}{\partial \psi_1} |_{\phi_0=\psi_0, \phi_1=\psi_1} < 1 \), which is equivalent to

\[
\bar{K} a^2 R^2 > \frac{2 - R}{R}
\]

The other partial derivative \( \frac{\partial T_1}{\partial \psi_0} |_{\phi_0=\psi_0, \phi_1=\psi_1} \) is

\[
\frac{\partial T_1}{\partial \psi_0} |_{\phi_0=\psi_0, \phi_1=\psi_1} = \frac{(\gamma_1 + \gamma_3) - \gamma_3 \gamma_6 \phi_1}{1 - (\gamma_6 + \gamma_2 \gamma_5) \phi_1} |_{\phi_0=\psi_0, \phi_1=\psi_1}
\]

\[
= \frac{(\bar{K} a^2 R^2 + 1)}{2} - \frac{1}{\bar{R}} \left( \frac{\bar{K} a^2 R^2 + 1}{2} - \frac{\bar{K} a^2 R^2 - 1}{2} \right) - \frac{\bar{K} a^2 R^2 - 1}{2R}
\]

The E-stability for the MSV RE equilibrium requires that \( \frac{\partial T_1}{\partial \psi_0} |_{\phi_0=\psi_0, \phi_1=\psi_1} < 1 \), which is equivalent to

\[
\bar{K} a^2 R^2 > \frac{2 - R}{R}
\]

To sum up, the E-stability for the MSV RE equilibrium requires that \( \bar{K} a^2 R^2 > \frac{2 - R}{\bar{R}} \).

The stationarity of the borrower’s collateral holding process (35) at REE requires

\[
\left| \frac{\gamma_5 \Psi_1}{1 - \gamma_6 \Psi_1} \right| < 1
\]

which is equivalent to \( \bar{K} a^2 R^2 > 1 \).
Figure 1: Responses to 1% unexpected temporary positive technology shock
Table 1: Dependence of the average defaulting time of borrowers on borrowers’ gain parameters and the size of the productivity shock

Panel A: Borrowers have smaller gain parameters

<table>
<thead>
<tr>
<th>STD</th>
<th>0.001</th>
<th>0.002</th>
<th>0.003</th>
<th>0.004</th>
<th>0.005</th>
<th>0.006</th>
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Panel B: Borrowers have larger gain parameters

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<th>0.017</th>
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<th>0.023</th>
<th>0.026</th>
<th>0.029</th>
<th>0.032</th>
<th>0.035</th>
<th>0.038</th>
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</thead>
</table>

The average defaulting time is calculated for different combinations of gain parameters in the learning rule of borrowers and different sizes of the productivity shock based on 5000 repetitions. The lenders’ gain parameter is set to 0.01. The standard deviation of the productivity shock (column STD) ranges from 0.1 to 1 multiple of 0.00712. For each repetition, the defaulting time is recorded as the first period when the expectation of borrowers is smaller than that of lenders.
Table 2: Dependence of the average defaulting time of borrowers on borrowers’ gain parameter and the size of the productivity shock: removing immediate default

Panel A: Borrowers have smaller gain parameters

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Panel B: Borrowers have larger gain parameters

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Note: compared with table 1, the only difference here is that immediate defaults are excluded when calculating the average default time.