Coarse Correlated Equilibria in an Abatement Game

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April 2013

Abstract

We consider coarse correlated equilibria – CCE – (Moulin and Vial 1978) for the well-analyzed
abatement game (Barrett 1994) and prove that CCE can strictly improve upon the Nash equilibrium
payoffs, while correlated equilibrium – CE – (Aumann 1974, 1987) cannot, because these games are
potential games. We compute the largest feasible total utility in any CCE in those games: it is
achieved by a CCE involving only two pure strategy profiles; however, the efficiency gain is small.

Keywords: Coarse correlated equilibrium, Abatement game.

JEL Classification Numbers: C72, Q52.
1 INTRODUCTION

We analyze the performance of coarse correlation in a well-studied non-cooperative model from the literature in environmental economics. The model, called the abatement game (Barrett 1994), is a game played by several players (nations) choosing the level of abatement (pollution). Although several (non-cooperative and cooperative) solutions have already been analyzed for the abatement game, the impact of strategic correlation has not. We submit that the concept of coarse correlation has a very natural interpretation in this game.

A correlation device is a lottery over the outcomes (strategy profiles) of the game. A correlated equilibrium (Aumann, 1974, 1987; thereafter CE) is implemented by a mediator who selects strategy profiles according to a publicly known probability distribution and sends to each player the private recommendation to play the corresponding realized strategy. The equilibrium property is that each player finds it optimal to follow this recommendation. In a coarse correlated equilibrium (Moulin and Vial 1978; thereafter CCE), the mediator requires more commitment from the players: it asks the players, before running the lottery, to either commit to the future outcome of the lottery or play any strategy of their own without learning anything about the outcome of the lottery. The equilibrium property is that each player finds it optimal to commit ex ante to use the strategy selected by the lottery.

In the context of climate change negotiation and in particular for the abatement game, a correlation device can be interpreted as an independent agency providing a recommendation to all relevant countries towards the ultimate goal of global emission reduction. In a CCE of the abatement game, each country remains free to revert to a non-cooperative emission, but does not benefit from doing so as long as other countries commit to the policy selected by the agency. Correlation, in either the CE or the CCE format, has been mostly ignored by the environmental literature.

It was recently discovered that CE cannot help improving upon the Nash equilibrium in many important microeconomic games. Liu (1996) and Yi (1997) proved that the only correlated equilibria in a large class of oligopoly games are mixtures of pure Nash equilibria, a result later on generalized by Neyman (1997) and Ui (2008) to all potential games with smooth and concave potential functions.

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1 Barrett (2001) and McGinty (2007) studied asymmetric versions of the abatement game. Barrett (1994) also considered the Stackelberg model of abatement which was later analyzed by Rubio and Ulph (2006). Finns (2001) presented generalization of Barrett’s results in terms of the number of countries in a stable equilibrium.

2 However, not fully, as shown by Kar, Ray and Serrano (2010).

3 In their paper, Moulin and Vial (1978) called this equilibrium concept a correlation scheme. Young (2004) and Roughgarden (2009) introduced the terminology of coarse correlated equilibrium that was later adopted by Ray and Sen Gupta (2013) and Moulin, Ray and Sen Gupta (2013), while Forgó (2010) called it a weak correlated equilibrium.

4 Forgó, Fülöp and Prill (2005) and Forgó (2011) recently used (modified versions of) Moulin and Vial’s notion of (coarse) correlation in other environmental games. Baliga and Maskin (2003) surveyed some models of mechanisms in this literature.
The abatement game is also a smooth potential game and hence its only CE is the (unique) Nash equilibrium. However, the game has many more CCEs; in some cases that we identify below, some of these are strictly more efficient than the Nash equilibrium outcome. We apply the general methodology introduced in Moulin, Ray and Sen Gupta (2013) to compute the most efficient CCE in a symmetric 2-person game with quadratic payoff functions. This (welfare) optimal CCE is a symmetric mixture of two pure strategy profiles.

Consider the following numerical example where player $i$ ($i = 1$ and $2$) chooses a non-negative $q_i$ with payoffs

$$u_1(q_1, q_2) = (q_1 + q_2) - 2(q_1 + q_2)^2 - q_1^2; u_2(q_1, q_2) = u_1(q_2, q_1).$$

The Nash equilibrium for this game is $(\frac{1}{10}, \frac{1}{10})$, with corresponding payoff (for any player) of $\frac{11}{100} = 0.11$.

Now consider the lottery

$$L = \frac{1}{2} \delta_{(z, z')} + \frac{1}{2} \delta_{(z', z)}$$

where $\delta_z$ is the deterministic outcome $z$, and where $z, z' = \frac{11+\sqrt{3}}{104}$. That is, with equal probability, one country abates $q_i = \frac{11+\sqrt{3}}{104}$ while the other chooses $\frac{11-\sqrt{3}}{104}$.

The above lottery is clearly not a CE because $(\frac{11+\sqrt{3}}{104}, \frac{11-\sqrt{3}}{104})$ is not a Nash equilibrium. But it is a CCE: if player 1 chooses $q_1$ and assumes player 2 is choosing either $\frac{11+\sqrt{3}}{104}$ or $\frac{11-\sqrt{3}}{104}$ with equal probability, his expected payoff $[\frac{11}{104}q_1 - 3q_1^2 + \frac{11}{104} - (\frac{11+\sqrt{3}}{104})^2 - (\frac{11-\sqrt{3}}{104})^2]$ is maximized at $q_1 = \frac{5}{2}$ and gives him $u_1 = \frac{299}{200}$, precisely the same as by committing to follow the outcome of $L$, that generates the expected utility of $u_1(L) = (z + z') - 2(z + z')^2 - \frac{1}{2}(z^2 + z'^2) = \frac{299}{200} \approx 0.1105$.

More importantly, we prove below that this lottery is actually the optimal CCE, with the total payoff

$$\pi^{CC} = 2u_1(L) = \frac{598}{200} = \frac{23}{10} \approx 0.2211.$$ 

One may be surprised to note that this CCE has an improvement ratio $\frac{\pi^{CC}}{\pi^{CE}}$ of only $\frac{525}{572} \approx 0.9052$, yielding just about $1\%$ increase over and above the Nash equilibrium payoff. This is because in this class of games, the Nash equilibrium can be actually very close to the efficient outcome (which maximizes the joint payoff). For this particular example, the efficient (total) payoff that the players can jointly achieve is $\frac{2}{5} \approx 0.2222$, and the Nash equilibrium (total) payoff $0.22$ is $99\%$ efficient. So the optimal CCE only incurs about $\frac{1}{2}\%$ of efficiency loss.

We formally characterize the optimal CCE for any 2-player abatement game (Theorem 1) under the assumption that the benefit parameter ($b$) is bigger than the cost parameter ($c$), after showing that the inequality $b > c$ is necessary to allow any improvement at all.\footnote{Gerard-Varet and Moulin (1978) proved that Nash equilibrium can be locally improvable by using a concept similar to CCE under a condition, which for this game, perhaps not surprisingly, also turns out to be $b > c$.} As in Moulin, Ray and Sen Gupta (2013), we find that the optimal lottery is a 2-dimensional anti-diagonal symmetric lottery.\footnote{As in the example, the optimal lottery is a symmetric lottery with finite support similar to those studied by Ray and Sen Gupta (2013) who called such a lottery a Simple Symmetric Correlation Device (SSCD) as introduced in Ganguly and Ray (2005) to discuss correlation.}
payoff at the optimal CCE is very close to the efficient payoff for this class of games, with a small
improvement above the Nash equilibrium total payoff.

We define CCEs in Subsection 2.1 and present the two-person abatement game in Subsection 2.2.
Section 3 develops our general methodology to compute the optimal CCE for the 2-player abatement
game while Section 4 concludes.

2 MODEL

2.1 Coarse Correlation

This subsection, borrowed from Moulin, Ray and Sen Gupta (2013), is given here for the sake of
completeness.

Fix a two-person normal form game, $G = [X_1, X_2; u_1, u_2]$, where the strategy sets are closed real
intervals and the payoff functions $u_i : X_1 \times X_2 \to \mathbb{R}$, $i = 1, 2$, are continuous. We write $C(X_1 \times X_2)$
for the set of such continuous functions and similarly, $C(X_i)$ for the set of continuous functions on $X_i$.

Let $\mathcal{L}(X_1 \times X_2)$ with generic element $L$ and $\mathcal{L}(X_i)$ with generic element $\ell_i$ be the sets of probability
measures on $X_1 \times X_2$ and $X_i$ respectively. That is to say, $L$ is a Radon measure$^7$ on $X_1 \times X_2$ of mass 1
(similarly, so is $\ell_i$, on $X_i$). We write the mean of $u_i(x_1, x_2)$ with respect to $L$ as $u_i(L)$; similarly, $f(\ell_i)$
is the mean of $f(x_i)$ with respect to $\ell_i$ for any $f \in C(X_i)$. Given $L \in \mathcal{L}(X_1 \times X_2)$, we write $L^1$ for the
marginal distribution of $L$ on $X_1$, defined as follows:

$$\forall f \in C(X_1), f(L^1) = f^*(L),$$

(with a symmetric definition for $L^2$).

The deterministic distribution at $z$ is denoted by $\delta_z$, and for a product distributions such as $\delta_{x_1} \otimes \ell_2$
we write $u_i(\delta_{x_1} \otimes \ell_2)$ simply as $u_i(x_1, \ell_2)$.

Definition 1 A coarse correlated equilibrium (CCE) of the game $G$ is a lottery $L \in \mathcal{L}(X_1 \times X_2)$ such
that

$$u_1(L) \geq u_1(x_1, L^2) \text{ and } u_2(L) \geq u_2(L^1, x_2) \text{ for all } (x_1, x_2) \in X_1 \times X_2. \quad (2)$$

The discussion of the mediator in the Introduction can be formalized by the normal form game
$\tilde{G} = [X_1 \cup \{\kappa\}, X_2 \cup \{\kappa\}; \tilde{u}_1, \tilde{u}_2]$ where the additional strategy $\kappa$ means “commit to the outcome of $L$”.
Payoffs are the same as in $G$ if no player chooses $\kappa$, otherwise are as follows:

$$\tilde{u}_1(\kappa, \kappa) = u_1(L); \tilde{u}_1(\kappa, x_2) = u_1(L^1, x_2); \tilde{u}_1(x_1, \kappa) = u_1(x_1, L^2)$$

(with a symmetric definition for $L^2$). Then, $L$ is a CCE if and only if $(\kappa, \kappa)$ is a Nash equilibrium of $\tilde{G}$.

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$^7$A positive linear functional on $C(X_1 \times X_2)$. 

4
2.2 Abatement Game and CCEs

We consider the model proposed in Barrett (1994) with two countries \((n = 2)\). The payoff function of a country is a function of the emission abated by both countries \(q_1\) and \(q_2\). Let us write the total emission abated as \(Q (Q = q_1 + q_2)\) and therefore we have the Benefit function\(^8\) of country \(i\) as

\[
B_i(Q) = \frac{B}{2}(AQ - \frac{Q^2}{2}).
\]

The cost function of each country is a function of its own emission abatement level \(q_i\) and is given as \(C_i(q_i) = \frac{Cq_i^2}{2}\). The payoff function of country 1 (and similarly for country 2) is thus given by

\[
u_1(q_1, q_2) = \frac{AB}{2}(q_1 + q_2) - \frac{B}{4}(q_1 + q_2)^2 - \frac{C}{2}q_1^2,\]
where \(A, B\) and \(C\) are all positive.

We call the above game an abatement game.

We set \(a = \frac{AB}{2}, b = \frac{B}{4}, c = \frac{C}{2}\) and rewrite the above payoff function in the following form (as in the general model of Moulin, Ray and Sen Gupta 2013):

\[
u_1(q_1, q_2) = a(q_1 + q_2) - b(q_1 + q_2)^2 - c\theta_1^2; \quad u_2(q_1, q_2) = u_1(q_2, q_1).
\]

Given \(q_2\), the best response of country 1 (and similarly for country 2) is \(BR_1(q_2) = \frac{\partial u_1(q_1, q_2)}{\partial \theta_1} = a - 2b(q_1 + q_2) - 2c\theta_1\). Therefore, the Nash equilibrium \((q_1^{Neq}, q_2^{Neq})\) and corresponding (total) payoff \(\pi^{Neq}\) are

\[
q_1^{Neq} = q_2^{Neq} = \frac{a}{2(2b + c)}; \quad \pi^{Neq} = \frac{a^2(4b + 3c)}{2(2b + c)^2}.
\]

We now compute the efficient profile of emission abatements \((q_1^{eff}, q_2^{eff})\). To maximize the total payoff \(u_1(q_1, q_2) + u_2(q_1, q_2) = 2a(q_1 + q_2) - 2b(q_1 + q_2)^2 - c(q_1^2 + q_2^2)\), we clearly need to choose \(q_1 = q_2\); we find

\[
q_1^{eff} = q_2^{eff} = \frac{a}{4b + c}; \quad \pi^{eff} = \frac{2a^2}{4b + c}.
\]

Therefore, the relative efficiency ratio of the Nash outcome is \(\frac{\pi^{N_{eq}}}{\pi^{eff}} = \frac{(4 \pm 3 \lambda)(4 + \lambda)}{\pi^{2 + \lambda}}\), where \(\lambda = \frac{a}{b}\), which is plotted below.

\(^8\)Note that the benefit function in the published version of Barrett (1994) has a typo that we have corrected here.
The abatement game is a potential game for the potential function $P(q_1, q_2) = a(q_1 + q_2) - b(q_1 + q_2)^2 - c(q_1^2 + q_2^2)$, which is smooth and concave. Therefore, the only CE is the Nash equilibrium $q^{Neq}$ (Neyman 1997).

3 RESULTS

Our goal is to compute the CCEs that maximize the total payoff $u_1 + u_2$ and to compare this joint payoff with the efficient payoff and with the Nash equilibrium payoff. As the abatement game is symmetric, we can limit our search to symmetric lotteries $L$ only (as explained in Moulin, Ray and Sen Gupta 2013, when we identify an optimal symmetric CCE, we are also capturing an optimal CCE among all CCEs, symmetric or otherwise). We denote the set of symmetric lotteries by $L_{sym}(\mathbb{R}_+^2)$.

We first characterize the equilibrium condition (2) presented in Definition 1 in terms of three moments of $L$. If $L$ is the distribution of the symmetric random variable $(Z_1, Z_2)$, these are respectively the expected values of $Z_1$, $Z_2^2$, and $Z_1 \cdot Z_2$ as denoted below.

$$\alpha = E_L[Z_1]; \beta = E_L[Z_1^2]; \gamma = E_L[Z_1 \cdot Z_2]$$

Lemma 1 A symmetric lottery $L \in L_{sym}(\mathbb{R}_+^2)$ is a CCE of the abatement game if and only if

$$\max_{z \geq 0} \{(a - 2\alpha)z - (b + c)z^2\} \leq a\alpha - (b + c)\beta - 2b\gamma$$

and the corresponding utility (for one player) is

$$u_1(L) = 2a\alpha - (2b + c)\beta - 2b\gamma.$$
Lemma 2 implies both lotteries with at most four strategy parameters.

The next result is due to Moulin, Ray and Sen Gupta (2013). It identifies the range of the vector \((\alpha, \beta, \gamma)\) when \(L \in \mathcal{L}^{sy}(\mathbb{R}_+^2)\) and also shows that this range is covered by two families of very simple lotteries with at most four strategy profiles in their support.

Let \(\mathcal{L}^*\) be the subset of \(\mathcal{L}^{sy}(\mathbb{R}_+^2)\) containing the simple lotteries of the form \(L = \frac{2}{3}(\delta_{z,z} + \delta_{z,z'}) + \frac{1}{3}(\delta_{z,z'} + \delta_{z',z}),\) where \(z, z', q\) and \(p\) are non-negative and \(q + p = 1\). Let \(\mathcal{L}^{**}\) be the subset of \(\mathcal{L}^{sy}(\mathbb{R}_+^2)\) of the form \(L = q \cdot \delta_{z,z} + q' \cdot \delta_{0,0} + \frac{2}{3}(\delta_{0,0} + \delta_{z,0}),\) where \(z, q, q'\) and \(p\) are non-negative and \(q + q' + p = 1\).

Lemma 2 i) For any \(L \in \mathcal{L}^{sy}(\mathbb{R}_+^2)\) and the corresponding random variable \((Z_1, Z_2)\), we have \(\alpha, \gamma \geq 0; \beta \geq \gamma; \beta + \gamma \geq 2\alpha^2;\) (5)

ii) Equality \(\beta = \gamma\) holds if and only if \(L\) is diagonal: \(Z_1 = Z_2\) (a.e.);

iii) Equality \(\beta + \gamma = 2\alpha^2\) holds if and only if \(L\) is anti-diagonal: \(Z_1 + Z_2\) is constant (a.e.);

iv) For any \((\alpha, \beta, \gamma) \in \mathbb{R}_+^2\) satisfying inequalities (5), there exists \(L \in \mathcal{L}^* \cup \mathcal{L}^{**}\) with precisely these parameters.

Note that (5) implies \(\beta \geq \alpha^2\), with equality \(\beta = \alpha^2\) if and only if \(L\) is deterministic, because \(\beta = \alpha^2\) implies both \(\beta = \gamma\) and \(\beta + \gamma = 2\alpha^2\).

Lemmata 1 and 2 now imply the following two-step algorithm (Proposition 1) to find the utility maximizing CCEs, which is similar to Theorem 1 in Moulin, Ray and Sen Gupta (2013).
Proposition 1 Given the abatement game, the following nested programs generate the utility maximizing CCEs:

Step 1: Fix $\alpha$ non negative, and solve the linear program

$$
\min_{\beta, \gamma} \{(2b + c)\beta + 2b\gamma\} \text{ under constraints }
$$

$$
\beta \geq \gamma \geq 0; \beta + \gamma \geq 2a^2; (b + c)\beta + 2b\gamma \leq \alpha \min_{z \geq 0} \{(a - 2b\alpha)z - (b + c)z^2\}.
$$

Step 2: With the solutions $\beta(\alpha), \gamma(\alpha)$ found in Step 1, solve

$$
\max_{\alpha} \{(2a\alpha - (2b + c)\beta(\alpha) - 2b\gamma(\alpha))) \text{ under constraints }
$$

$$
\alpha \geq 0; \max_{z \geq 0} \{(a - 2b\alpha)z - (b + c)z^2\} \leq \alpha - (b + c)\beta(\alpha) - 2b\gamma(\alpha).
$$

Moreover, there is a utility maximizing CCE in $L^* \cup L^{**}$.

3.1 Optimal CCE

Proposition 1 implies the following characterization of the utility maximizing CCE for our games.

Theorem 1 i) If $b \leq c$, the Nash equilibrium of the abatement game is its only CCE.

ii) If $b > c$, setting $\lambda = \frac{\alpha}{b}$, the optimal values of the three moments of the utility maximizing $L$ are given by $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$:

$$
\tilde{\alpha} = \frac{a^2 4 + 8\lambda^2 - 4\lambda^3}{b^2 4(4 + 5\lambda)^2},
$$

$$
\tilde{\beta} = \frac{a^2}{b^2} \frac{4 + 8\lambda + \lambda^2 - 4\lambda^3}{4(4 + 5\lambda)^2}, \text{ and } \tilde{\gamma} = \frac{a^2}{b^2} \frac{4 \lambda^2 - 4\lambda^3 + 2\lambda^4}{4(4 + 5\lambda)^2};
$$

while the optimal CCE is $\tilde{L} = \frac{1}{2} \delta(z, z') + \frac{1}{2} \delta(z', z)$, with

$$
z, z' = \frac{a^2 2 + 2\lambda - \lambda^2 \pm \lambda \sqrt{1 - \lambda^2}}{b^2 4(4 + 5\lambda)}.
$$

Proof. First, consider the equilibrium condition (4) as in Lemma 1. Note that if $a - 2b\alpha < 0 \iff \alpha > \frac{a}{2b}$, the L.H.S. of that inequality (the maximum over $z \geq 0$) is zero; therefore, (4) becomes

$$
\alpha a \geq (b + c)\beta + 2b\gamma = b(\beta + \gamma) + c\beta + b\gamma > b(\beta + \gamma) \geq 2a^2,
$$

which is a contradiction. So, we must have $\alpha \leq \frac{a}{2b}$; then the L.H.S. of (4) is $\frac{(a - 2b\alpha)^2}{4(b + c)}$. The equilibrium condition is now

$$
(b + c)\beta + 2b\gamma \leq \alpha a - \frac{(a - 2b\alpha)^2}{4(b + c)} = \frac{b^2 \alpha^2 - a(2b + c)\alpha + \frac{a^2}{b + c}}{b + c}.
$$

(6)

We now fix $\alpha$ and solve Step 1 in Proposition 1: we must minimize $(2b + c)\beta + 2b\gamma$ in the polytope $\Psi = \{(\beta, \gamma)|\beta \geq \gamma, \beta + \gamma \geq 2a^2\}$ under the additional constraint (6). Note that $\Psi$ is unbounded
from above and bounded from below by the interval \([P,Q]\), where \(P = (\alpha^2, \alpha^2)\) and \(Q = (2\alpha^2, 0)\). We distinguish two cases here.

**Case 1** \((b \leq c)\): In this case, the minimum in \(\Psi\) of both \((2b + c)\beta + 2b\gamma\) and \((b + c)\beta + 2b\gamma\) is achieved at \(P\). Therefore, if \(P\) meets (6) it is our optimal pair \((\beta(\alpha), \gamma(\alpha))\); otherwise, there is no CCE for this choice of \(\alpha\). Now, \(P\) meets (6) if and only if \((3b + c)\alpha^2 \leq -\frac{\beta^2 a^2 - a(2b + c)\alpha + \alpha^2}{b + c}\), which reduces to \([\alpha - (2b + c)\alpha]^2 \leq 0 \iff \alpha = \pm \frac{\alpha}{2(2b + c)} = q_{1i}^{Neq}\). By Lemma 2, the optimal CCE \(L\) is diagonal \((\beta = \gamma)\) and deterministic \((\beta = \alpha^2)\). It is simply the Nash equilibrium \(L = \delta_{q_{1i}^{Neq}}\) of our game.

**Case 2** \((b > c)\): Here, the minimum of \((b + c)\beta + 2b\gamma\) in \(\Psi\) is achieved at \(Q\); so, if \(Q\) fails to meet the constraint (6) there is no hope to meet it anywhere in \(\Psi\). Thus, we must choose \(\alpha\) such that

\[
2(b + c)\alpha^2 \leq -\frac{b^2\alpha^2 - a(2b + c)\alpha + \alpha^2}{b + c} \iff \Lambda(\alpha) = (3b^2 + 4bc + 2c^2)\alpha^2 - a(2b + c)\alpha + \frac{\alpha^2}{4} \leq 0 \quad (7)
\]

The discriminant of the right-hand polynomial \(\Lambda(\alpha)\) is \(a^2(b^2 - c^2)\); therefore, (7) restricts \(\alpha\) to an interval \([\alpha_-, \alpha_+]\), between the two positive roots of \(\Lambda(\alpha)\). For such a choice of \(\alpha\), the constraint (6) cuts a subinterval \([R, Q]\) of \([P, Q]\), where \(R\) meets (6) as an equality. Note that \(R = P\) only if \(\alpha = q_{1i}^{Neq}\) (from Case 1 and the fact that \(\Lambda(q_{1i}^{Neq}) < 0\)), otherwise \(R \neq P\). Clearly, \(R\) is our optimal choice \((\beta(\alpha), \gamma(\alpha))\) and it solves the system

\[
\beta + \gamma = 2\alpha^2; \quad (b + c)\beta + 2b\gamma = -\frac{b^2\alpha^2 - a(2b + c)\alpha + \alpha^2}{b + c}.
\]

Therefore,

\[
\beta(\alpha) = \frac{1}{b^2 - c^2} \left[ b(5b + 4c)\alpha^2 - a(2b + c)\alpha + \frac{\alpha^2}{4} \right] \quad \text{and}
\gamma(\alpha) = \frac{1}{b^2 - c^2} \left[ -(3b^2 + 4bc + 2c^2)\alpha^2 + a(2b + c)\alpha - \frac{\alpha^2}{4} \right].
\]

Now in Step 2 of Proposition 1, we must maximize \(2a\alpha - (2b + c)\beta(\alpha) - 2b\gamma(\alpha)\) under the constraints \(\alpha \geq 0\) and \(\Lambda(\alpha) \leq 0\). Developing this objective function yields the program

\[
\frac{1}{b^2 - c^2} \max_\alpha \left\{ -b^2(4b + 5c)\alpha^2 + a(2b^2 + 2bc - c^2)\alpha - \frac{\alpha^2 c}{4} \right\} \quad (8)
\]

under the constraints

\[
\alpha \geq 0 \quad \text{and} \quad \Lambda(\alpha) = (3b^2 + 4bc + 2c^2)\alpha^2 - a(2b + c)\alpha + \frac{\alpha^2}{4} \leq 0.
\]

The unconstrained maximum of the objective function is achieved at \(\tilde{\alpha} = \frac{a(2b^2 + 2bc - c^2)}{2b^2(4b + 5c)}\).

We now show that \(\Lambda(\tilde{\alpha}) \leq 0\). With the change of variable \(\lambda = \frac{\alpha}{\tilde{\alpha}}\), this amounts to

\[
\frac{(3 + 4\lambda + 2\lambda^2)(2 + 2\lambda - \lambda^2)^2}{4(4 + 5\lambda)^2} - \frac{(2 + \lambda)(2 + 2\lambda - \lambda^2)}{2(4 + 5\lambda)} + \frac{1}{4} \leq 0
\]

\[
\iff 4 + 8\lambda - 5\lambda^2 - 12\lambda^3 + 3\lambda^4 + 4\lambda^5 - 2\lambda^6 \geq 0
\]
The above polynomial is 0 at $\lambda = 1$; it is also easy to check, numerically, that it is non-negative on $[0,1]$. The proof of is complete if we now express $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ in terms of $\lambda$. This is indeed easy for $\tilde{\alpha}$. One may also verify, using the expression for $\tilde{\alpha}$ that

$$\tilde{\beta} = \beta(\tilde{\alpha}) = \frac{1}{b^2 - c^2} \left[ b(b(5b + 4c)\tilde{\alpha}^2 - a(2b + c)\tilde{\alpha} + \frac{a^2}{4} \right)$$

$$= \frac{a^2 4 + 8\lambda + \lambda^2 - 4\lambda^3}{4(4 + 5\lambda)^2}$$

and

$$\tilde{\gamma} = \gamma(\tilde{\alpha}) = \frac{1}{b^2 - c^2} \left[ -(3b^2 + 4bc + 2c^2)\tilde{\alpha}^2 + a(2b + c)\tilde{\alpha} - \frac{a^2}{4} \right]$$

$$= \frac{a^2 4 + 8\lambda - \lambda^2 - 4\lambda^3 + 2\lambda^4}{4(4 + 5\lambda)^2}.$$

Finally, we construct the optimal CCE $\tilde{L}$. From $\tilde{\beta} + \tilde{\gamma} = 2\tilde{\alpha}^2$ and Lemma 2(iii), we see that $\tilde{L}$ is an anti-diagonal lottery of the form $\tilde{L} = \frac{1}{4\beta(\tilde{z},\tilde{z}')} + \frac{1}{4\beta(\tilde{z}',\tilde{z})}$, where $\tilde{z}$ and $\tilde{z}'$ are non-negative numbers such that $\tilde{z} + \tilde{z}' = 2\tilde{\alpha}$ and $\tilde{z}^2 + \tilde{z}'^2 = 2\tilde{\beta}$. This implies $2\tilde{z}\tilde{z}' = (2\tilde{\alpha})^2 - (2\tilde{\beta}) = 2\tilde{\gamma}$, hence $z$, $z'$ solve $Z^2 - 2\alpha Z + \gamma = 0$. The discriminant is $\alpha^2 - \gamma = \beta - \alpha^2 = \frac{a^2 4 + 4\lambda - \lambda^2}{4(4 + 5\lambda)^2}$; thus the expressions for $z$ and $z'$ follow.

### 3.2 Efficiency Performance

We compare now the optimal CCE (total) profit $\pi_{CC} = 2u_1(\tilde{L})$, to both the efficient and the Nash equilibrium (total) profits. From the expression (8) of the single player profit $u_1(\tilde{L})$ and the expression of $\tilde{\alpha}$ in Theorem 1(ii), straightforward computations provide

$$u_1(\tilde{L}) = \frac{1}{b^2 - c^2} \frac{a^2 (2 + 2\lambda - \lambda^2)^2}{4(4 + 5\lambda)} - \frac{a^2 c}{4b^2} = \frac{a^2 4 + 4\lambda - \lambda^2}{b 4(4 + 5\lambda)}.$$

Recalling

$$\pi_{EF} = \frac{2a^2}{4b + c} = \frac{a^2}{b} \frac{2}{4 + \lambda}$$

and

$$\pi_{N} = \frac{a^2 (4b + 3c)}{2(2b + c)^2} = \frac{a^2}{b} \frac{4 + 3\lambda}{2(2 + \lambda)^2},$$

we can now state the following.

**Corollary 1** For the abatement game, the relative efficiency of the optimal CCE and its relative improvement over the symmetric Nash equilibrium payoff depend only upon $\lambda = \frac{c}{b}$, as follows:

$$\frac{\pi_{CC}}{\pi_{EF}} = \frac{(4 + \lambda)(4 + 4\lambda - \lambda^2)}{4(4 + 5\lambda)} \text{ for } 0 \leq \lambda \leq 1,$$

$$\frac{\pi_{CC}}{\pi_{N}} = \frac{(2 + \lambda)(4 + 4\lambda - \lambda^2)}{(4 + 5\lambda)(4 + 3\lambda)} \text{ for } 0 \leq \lambda \leq 1; \frac{\pi_{CC}}{\pi_{N}} = 1 \text{ for } \lambda \geq 1.$$

Corollary 1 on the behaviour of the efficiency ratios is described in Figures 2 and 3.
4 CONCLUSION

We have analyzed coarse correlated equilibria in a class of 2-person symmetric game called the abatement game where correlation a la Aumann does not offer anything more than the Nash equilibrium. Incorporating the techniques introduced by Moulin, Ray and Sen Gupta (2013), we have characterized the utility maximizing CCE and have shown that they have a very simple support with only four deterministic strategy profiles. Such a computation is the first of its kind for coarse correlated equilibria for the abatement game and, this is why we regard this exercise as an interesting first step towards more
sophisticated computations to understand mediation in general for such games.

We conclude by revisiting the example in the Introduction and illustrate our results more formally. Consider the following values of the parameters, $a = 1$, $b = 2$ and $c = 1$ in the abatement game. Here, $\lambda = \frac{c}{b} = \frac{1}{2} < 1$ and the payoff function is given by $u(q_1, q_2) = (q_1 + q_2) - 2(q_1 + q_2)^2 - q_1^2$, with Nash equilibrium quantity, $q^{Neq} = \frac{2a}{(2b + c)} = \frac{1}{10}$.

From Theorem 1, the corresponding optimal values of the moments are:

- $\tilde{\alpha} = \frac{11}{10} \approx 1.057$,
- $\tilde{\beta} = \frac{31}{2704} \approx 0.0114$ and
- $\tilde{\gamma} = \frac{59}{5408} \approx 0.0109$.

The optimal CCE is the lottery $\tilde{L} = \frac{1}{2} \delta(z, z') + \frac{1}{2} \delta(z', z)$, where $z, z' = \frac{11 \pm \sqrt{3}}{10}$. The corresponding expected utility (for one player) derived by playing this CCE is $u_1(\tilde{L}) = \frac{299}{2704} \approx 0.1105$.

The optimal CCE (total) payoff is $\pi^{CCE} = 2u_1(\tilde{L}) = \frac{299}{2704} \approx 0.2211$, while the efficient (total) payoff is $\pi^{eff} = \frac{2}{5} \approx 0.4 \approx 0.2222$ and the Nash equilibrium (total) payoff is $\pi^{Neq} = \frac{11}{54} \approx 0.22$ (and hence $\frac{\pi^{Neq}}{\pi^{eff}} = 0.99$).

Using Corollary 1, the corresponding efficiency ratios here are $\frac{\pi^{CCE}}{\pi^{eff}} = \frac{299}{207} \approx 0.9951$ and $\frac{\pi^{CCE}}{\pi^{Neq}} = \frac{575}{572} \approx 1.0052$. 

12
5 REFERENCES


