Learning Dynamics with Data (Quasi-) Differencing

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Abstract

The paper studies learning with data (quasi-)differencing where agents need to (quasi-)difference data and then use an otherwise standard least squares learning procedure. It (1) establishes that the \textit{E-stability Principle} is still valid for analyzing the convergence of the learning with (quasi-)differencing data process to the Rational Expectations Equilibrium (REE), (2) provides new perspectives on the stability of the REE, the stability of the Rational Expectations bubble solutions, and equilibrium selection under adaptive learning, (3) demonstrates the importance of considering agents’ uncertainty and learning about the long-run growth of endogenous variables in dynamic macroeconomic models, (4) provides recommendations and a caveat on addressing model misspecifications in econometric practice, and (5) shows learning with (quasi-)differencing data helps understand some salient features of fluctuations in asset prices, inflation and aggregate economic activities.

Keywords: Expectations, Convergence, Long-Run Growth, Serial Correlation, Bubbles, Underparameterization

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1 Introduction

A long strand of literature replaces the Rational Expectations (RE) assumption by adaptive learning, assuming economic agents act like econometricians when doing the forecasting

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about the future state of the economy.\footnote{Examples are Bray (1982), Marcet and Sargent (1989), Woodford (1990), Duffy (1994), Evans and Honkapohja (2001), Bullard and Mitra (2002), Adam (2003), Sargent, Williams and Zha (2006), and Eusepi and Preston (2011).} While the literature mainly considers least squares learning algorithm, the paper considers learning with (quasi-)differencing data by which agents need to quasi-difference or difference their data and then use an otherwise standard least squares learning procedure.

Estimating models with filtering or (quasi-)differencing data is widely present in the practice of macroeconomics, econometrics and forecasting. Examples of differencing data are Vector Autoregressive Integrated Moving Average (VARIMA) models, cointegration and error correction models, and trend-cycle decomposition (e.g., Beveridge-Nelson (BN) decomposition). Agents difference the data when finding it non-stationary. In addition, random walk beliefs can be nearly self-fulfilling under learning\footnote{See Branch and Evans (2011, 2013) for examples in asset pricing models and New Keynesian Models and also my examples in Section 2.2.} which may justify data differencing.

When agents are learning, serially correlated residuals can arise due to gradual adjustment of beliefs to a new equilibrium after a structural change or omission of serially correlated and possibly unobserved variables as regressors.\footnote{Bray and Savin (1986) is an example showing that serially correlated residuals arise when agents are learning.} Agents will use a special case of the Generalized Methods of Moments, i.e., (Feasible) Generalized Least Squares ((F)GLS) and estimate models with quasi-differencing the data when being alert to potential model misspecifications or allowing for a richer structure of the variance-covariance matrix of the residuals which invalidates the optimality of the ordinary least squares.

The paper derives the recursive formulation of the learning with (quasi-)differencing data algorithms, enabling the study of a number of issues in adaptive learning. The \textit{E-stability Principle}, i.e., the correspondence between the Expectational Stability (E-stability) of a REE and its stability under adaptive learning, is an operating hypothesis for analyzing the convergence and stability under adaptive learning. Marcet and Sargent (1989) show that the correspondence holds for least squares learning in a general class of linear stochastic models. I show that the \textit{E-stability Principle} is still valid for analyzing the convergence of the learning with (quasi-)differencing data process. Nevertheless, the stability condition of a REE under learning with (quasi-)differencing data can be identical or different from the stability condition of the REE under least squares learning, depending on the model class considered and on whether agents learn the persistence of the residuals or not.

Least squares learning of a misspecified and underparameterized model usually converges to a restricted perception equilibrium.\footnote{See Chapter 3 of Evans and Honkapohja (2001) for an example.} For the first time in the literature, I establish the
convergence of learning underparameterized models to a REE. The convergence becomes possible by considering learning with (quasi-)differencing data and noting that the linear RE solution for a serially correlated variable can be expressed equivalently as an underparameterized model plus serially correlated residuals. I show that the convergence condition can be obtained by the \textit{E-stability principle} and is tighter than the E-stability condition of the REE under least squares learning in an example.

I argue that analyzing the convergence of learning misspecified models has important implications for addressing model misspecifications in econometric practice. As in the example, to address the underparameterization, adjusting the model specification is more desirable and recommended than using more sophisticated estimation methods because the latter involves estimating some extra parameters and imposes a stronger condition to ensure the convergence to the REE. Put differently, instability of the REE or divergence of the estimates happens more likely in the latter.

Differencing (possibly the logarithm of) the data is routinely performed in macroeconomics, such as to calculate the growth rate of a variable and to obtain the trend/cyclical component by the BN decomposition.\textsuperscript{5} An important reason why data differencing has not attracted sufficient attention in the adaptive learning literature is that the existing business cycle models with adaptive learning consider only learning about detrended variables, implying that agents are endowed with the exact knowledge about the long-run growth of endogenous variables by modelers.\textsuperscript{6} I show that the condition governing the convergence of the process with learning about the (long-run) growth of endogenous variables is looser than the E-stability condition when learning about detrended variables in an example and that data differencing helps understand excess volatilities of stock returns and macroeconomic activities.

Adopting more sophisticated methods (e.g., (F)GLS) than Ordinary Least Squares (OLS) implies that the econometric specification is flexible and more general such that it can nest the RE equilibria with different functional forms (say Minimal State Variable (MSV) solution\textsuperscript{7} and non-MSV solution) in models with multiple RE equilibria. In a model class, an example of which is an overlapping generations model with fiat money, the paper studies the convergence of learning with (quasi-)differencing process to the RE equilibria and derives the

\textsuperscript{5}Data differencing also relates to the HP filter. Harvey and Jeager (1997) show that extracting the trend by discounted least squares estimation of the growth rate or the mean of the differenced data is a special case of the HP filter by choosing the smooth parameter in a particular way, see also Chapter 3 of Canova (2007).

\textsuperscript{6}See Kuang and Mitra (2014) for further discussion.

\textsuperscript{7}The concept of the MSV solution was introduced in McCallum (1983) and is a solution which depends linearly on a set of variables and which is such that there does not exist a solution which depends linearly on a smaller set of variables.
associated convergence condition, which can give a prediction on the equilibrium selected by
the more sophisticated methods.

Whether adaptive learning can rule out rational bubble solutions has been studied by Evans (1989), McCallum (2009a, 2009b), Cochrane (2009, 2011) and Evans and McGough (2010) among others, in asset pricing models or New Keynesian models of inflation. The literature performs a E-stability analysis associated with learning an explosive process. Noting the long-run growth rate of a bubbly variable (say stock prices) is constant in the RE bubble solution. The paper considers an alternative expectation formation method: learning about the growth rates of the bubbly variable. It is shown that agents’ long-run growth beliefs will converge to the RE value but the rest point of the learning process is instable, reinforcing the point of McCallum (2009a, 2009b) and Evans and McGough (2010) that bubble solutions are fragile.

The paper shows that beliefs about persistent residuals (including random walk beliefs as a special case) or data (quasi-)differencing can be nearly self-fulfilling. Suppose agents estimate models with (quasi-)differencing data perceiving persistent residuals, for example, in a basic Real Business Cycle model or New Keynesian model. The resulting Actual Law of Motion (ALM) under learning for endogenous variables can be very close to their perception. In addition, the persistence and volatilities of endogenous variables (e.g., inflation, capital) are an increasing function of agents’ perceived persistence of residuals $\rho$: displaying higher (lower) persistence and volatilities relative to RE when $\rho$ is larger (smaller) than the corresponding RE value. A numerical example suggests that learning $\rho$ over time and estimating models with filtering data can be used to explain endogenous drifting and regime switching in persistence and volatilities of macroeconomic activities.

I show that data (quasi-)differencing and Generalized Least Squares (GLS) estimation can be exactly self-fulfilling. When agents perceive or detect serially correlated residuals, GLS estimation is recommended by econometricians. The residual persistence in the ALM under learning can be identical to agents’ (wrong) subjective perception which justifies the utilization of GLS. A simple check by agents, i.e., calculating the persistence of the residuals cannot detect the problem that they get the wrong estimates relative to the RE value. Either an alternative specification of the model or careful diagnosis is needed to detect the problem in order to learn the REE.

The rest of this paper is structured as follows. Section 2 (3) presents the convergence of learning with data (quasi-)differencing when regressors contain only current period exogenous variables (lagged and stochastic variables) and numerical examples. Section 4 examines learning with estimating residual persistence and (quasi-)differencing data. Section 5 considers equilibrium selection in a different class of models. Section 6 concludes.
2 Learning with Data (Quasi-)Differencing

Consider the following model of $y_t$

\[ y_t = \alpha E_t y_{t+1} + \delta x_t + v_t \]  
\[ x_t = dx_{t-1} + \xi_t \]

where $\{v_t\}$ is an identically and independently distributed (i.i.d.) process with zero mean and constant variance $\sigma_v^2$. $\{\xi_t\}$ is an i.i.d. process with zero mean and constant variance and independent to the disturbances $\{v_t\}$. $x_t$ contains only non-stochastic variables and/or current period exogenous and stochastic variables in this section and contains lagged and stochastic variables (either endogenous or exogenous) in section 3. The model can be viewed as a risk-neutral asset pricing model or a hyperinflationary model.\(^8\) $\alpha d \neq 1$ is assumed because $\alpha d = 1$ causes a singularity in the T-map mapping agents’ subjective belief to the parameter in the actual law of motion as can be seen later. The usual assumption on $d$ is $|d| < 1$. In the paper this is extended to include $d = 1$, which corresponds to that $x_t$ is non-stationary and contains a unit root.

2.1 Perceived Law of Motion (PLM)

Suppose agents perceive the residuals in their regression models to be serially correlated say due to omission of persistent and possibly unobserved variables or during the transition to the REE. For the ease of exposition, the residuals are assumed to follow a first-order Autoregressive (AR(1)) process.\(^9\)

Specifically, agents’ perceived law of motion (PLM)

\[ y_t = ax_t + w_t \]  
\[ w_t = \rho w_{t-1} + \eta_t \]

where $\eta_t$ is i.i.d distributed and its variance, $\sigma_\eta^2 = E(\eta_t^2)$, is normalized to 1. The case with $\rho = 0$ corresponds to the usual case in which agents apply OLS to learn the unknown parameter $a$. When $\rho = 1$, agents will estimate the model parameter $a$ with differencing data. When $|\rho| < 1$, agents will apply GLS.

In this section, agents are assumed not to update their perceived persistence of residuals

\(^8\)When $v_t$ in (1) is set to zero for all time $t$, the model (1)-(2) is the same as model d) of Marcet and Sargent (1989).

\(^9\)It is straightforward to extend to the case with a richer structure of the variance-covariance matrix of residuals, such as higher order serial correlation of residuals.
The case without updating $\rho$ is interesting for several reasons. First, an important special case that agents do not update $\rho$ is $\rho = 1$, when agents perceive a stochastic trend or learn about the growth rates and difference (possibly the logarithm of) the data. Second, agents may not update their beliefs about $\rho$ when they are nearly or exactly self-fulfilling under learning; see Example 3.1, 3.2 and Section 3.2. Third, not updating $\rho$ can be viewed as a limiting case that agents update their perceived persistence very slowly, which serves as a benchmark for understanding the dynamics when agents learn $\rho$ over time studied in section 4 and 5.

2.2 Current Period Exogenous Variables as Regressors

2.2.1 REE and The E-Stability Principle

Consider the REE with the form $y_t = ax_t + bv_t$. Given agents’ estimates $a_t$ and $b_t$, conditional expectations are $E_t y_{t+1} = a_t E_t x_{t+1} = a_t dx_t$. Plugging the conditional expectations into model (1) yields the ALM

$$y_t = T^O(a_t)x_t + v_t$$

where $T^O(a_t) = a_t d + \delta$, i.e., the T-map which maps agents’ subjective estimates $a_t$ to the parameter in the ALM. The RE values of $a$ and $b$ are $a^{RE} = a_t = \frac{\delta}{1 - \alpha d}$ and $b^{RE} = b_t = 1, \forall t$.

The correspondence between E-stability of an REE and its stability under adaptive learning is called the E-stability Principle, which is an operating hypothesis for analyzing the convergence and stability under adaptive learning. The E-stability is determined by the stability of an Ordinary Differential Equation (ODE)

$$\frac{\partial \phi}{\partial \tau} = T(\phi) - \phi$$

where $\tau$ denotes notional time, $\phi$ agents’ subjective estimate, and $T(\phi)$ the mapping which maps $\phi$ to the corresponding parameter in the ALM under adaptive learning.

Marcet and Sargent (1989) show the correspondence between the Expectational Stability of a REE and its stability under least squares learning in a general class of linear stochastic models. According to the E-stability principle, the stability of the REE under least squares learning is determined by the E-stability condition, i.e., the stability of (5) with $\phi = a$ and $T(\phi) = T^O(a)$. The E-stability condition is $\alpha d < 1$.

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10 Here agents are assumed to know the persistence coefficient of $x_t$. Alternatively, agents can have the PLM relating $y_t$ to $x_{t-1}$ rather than $x_t$.

2.2.2 Learning with Data (Quasi-)Differencing (LDD)

Suppose agents hold the subjective beliefs (3)-(4) with \( \rho \neq 0 \) and learn about \( a \) over time using past data. The parameter \( a \) can be estimated via two steps. The data are filtered or (quasi-)differenced in the first step and OLS is applied to the (quasi-)differenced data in the second. Transforming agents’ model (3) – (4) yields

\[
y_t - \rho y_{t-1} = a(x_t - \rho x_{t-1}) + \eta_t
\]

for \( \forall t \geq 2 \). Model (6) can be estimated via OLS because agents perceive \( \eta_t \) to be i.i.d.

Following the standard assumption in the literature, agents’ belief at period \( t \) is determined by the data up to time \( t-1 \) to avoid the simultaneous determination of the data and agents’ beliefs. The recursive formulation of the estimator\(^{12}\)

\[
a_t = a_{t-1} + \frac{1}{t} S_{t-1}^{-1}(x_{t-1} - \rho x_{t-2})[(y_{t-1} - \rho y_{t-2}) - (x_{t-1} - \rho x_{t-2})a_{t-1}]
\]

\[
S_t = S_{t-1} + \frac{1}{t}[(x_t - \rho x_{t-1})^2 - S_{t-1}]
\]

given \( y_0, y_{-1}, x_0, x_{-1}, a_0, S_0 \). The usual Recursive Least Squares formula is applied but to (quasi-)differenced data. Learning with data (quasi-)differencing is defined as follows.

**Definition 2.1 Learning with data (quasi-)differencing (LDD)**\(^{13}\)

Agents (quasi-)difference their data with their perceived persistence of residuals \( \rho \) and then apply an otherwise standard recursive least squares learning procedure to the (quasi-)differenced data.

Leading equation (6) for one period and taking conditional expectations yield \( E_t y_{t+1} = \rho y_t + a_t d - a_t \rho x_t \). Note the conditional expectations under LDD differ from the counterpart under least squares learning, i.e., \( a_t dx_t \). The difference arises since agents try to improve their forecasts using their perception of serially correlated residuals.

Plugging the expectations into model (1) yields the ALM under LDD

\[
y_t = T_1(a_t; \rho)x_t + T_2(a_t; \rho)v_t
\]

where \( T_1(a_t; \rho) = \frac{\delta - \alpha a_t \rho - a_t d}{1 - \alpha \rho} \) and \( T_2(a_t; \rho) = \frac{1}{1 - \alpha \rho} \) assuming \( \alpha \rho \neq 1 \). Carrying from the

\(^{12}\)It can be shown that this recursive learning with (quasi-)differencing data algorithm is optimal and can be derived from Bayesian updating given a set of subjective beliefs about the data.

\(^{13}\)Readers may wonder why the learning algorithm is not called “Generalized Least Squares Learning.” The GLS estimator typically considers the case the persistence of residuals \( \rho \) is in the interval \((-1, 1)\). In the paper \( \rho \) is in general not restricted in this interval, say \( \rho = 1 \) is allowed.
conditional expectations, the T-map under LDD depends not only on agents’ estimate of \( a_t \), but also on \( \rho \).

2.2.3 Convergence Results

The following proposition summarizes the convergence result for the learning process. Denote \( a^* \) the rest point of the LDD process.

Proposition 2.1

(a) Agents’ belief \( a_t \) under LDD will converge to the RE value \( \frac{\delta}{1-\alpha d} \) under the conditions

\( (1) \) \( \alpha d < 1 \) and \( \alpha \rho < 1 \); or \( (2) \) \( \alpha d > 1 \) and \( \alpha \rho > 1 \).

(b) At the rest point, the parameter in the ALM \( T_1(a^*; \rho) \) equals agents’ belief \( a^* \), i.e., \( T_1(a^*; \rho) = a^* = \frac{\delta}{1-\alpha d} \). If \( 0 < \alpha < 1 \) and \( \rho < \alpha^{-1} \), the volatilities of \( y_t \) are an increasing function of \( \rho \) and higher (lower) than those under RE if \( \rho \) is larger (smaller) than the corresponding RE value 0.

Proof. See Appendix A. □

Part (a) of the proposition consists of two statements. First, the belief \( a_t \) under LDD will still converge to the RE value. The intuition is as follows. Fixing agents’ belief \( a_t \) at \( a \) and using the ALM (9) to replace \( y_t \) and \( y_{t-1} \) in agents’ model (6) yield

\[ y_t - \rho y_{t-1} = T_1(a; \rho)(x_t - \rho x_{t-1}) + T_2(a; \rho)(v_t - \rho v_{t-1}) \]

The transformed residuals, \( v_t - \rho v_{t-1} \), are uncorrelated with the transformed regressor \( x_t - \rho x_{t-1} \). Quasi-differencing the data with incorrectly perceived persistence \( \rho \neq 0 \) does not generate a correlation between the transformed regressors and the residuals in the second step OLS regression due to the exogeneity of the transformed regressors. So agents can still get an unbiased estimate, i.e., the RE value, when applying OLS to model (6).

This result is analogous to a result in classical econometrics when regressors are exogenous: if residuals are serially correlated and agents use OLS to estimate the parameters in their model, they still get an unbiased estimate. In the current context, if true disturbances are not serially correlated under RE and learning agents apply the estimator with data (quasi-)differencing due to the perception or detection of serially correlated residuals, they still get an unbiased estimate due to the exogeneity of the regressors.

Appendix A shows that the stability of the ODE (5) where \( \phi = a \) and \( T(\phi) = T_1(a) \) defined just after equation (9) determines the stability condition of the rest point under LDD stated in Part (a) of the proposition. The E-stability Principle used to analyze the convergence of the learning process to the RE is still valid and useful for analyzing the
convergence of the LDD process if the learning process converges to the fixed point of the T-map.\(^\text{14}\) Note the convergence condition in part (a) is different from the E-stability condition of the REE under least squares learning because the T-map and hence the stability condition depend on \(\rho\).

Part (b) can be verified by substituting the RE value \(a^{RE}\) into \(T_1\) and \(T_2\)-map. This yields \(T_1(a^{RE}; \rho) = a^{RE} = \frac{\delta}{1-\alpha d}\) and the volatilities of \(y_t\) are increasing in \(\rho\) because \(T_2(a^{RE}; \rho)\) is increasing in \(\rho\) when \(0 < \alpha < 1\) and \(\rho < \alpha^{-1}\).

The corollary below emphasizes two interesting special cases.

**Corollary 2.1**

(a) In the case that \(x_t\) contains a stochastic trend \(d = 1\), for all parameterization \(\alpha \neq 1\) agents’ belief \(a_t\) under LDD (i.e., when agents’ PLM is \(\Delta y_t = a \Delta x_t + \eta_t\)) will converge to the RE value \(\frac{\delta}{1-\alpha}\).

(b) If agents are endowed with the knowledge about the persistence of \(x_t\) and hold the belief \(\rho = d\), then agents’ belief \(a_t\) under LDD will converge to the RE value \(\frac{\delta}{1-ad}\) under all parameterization \(ad \neq 1\).

The E-stability condition of the REE under least squares learning appears to have a discontinuity at \(d = 1\) where the underlying driving process contains a stochastic trend \(d = 1\). On the one hand, the E-stability condition is \(ad < 1\). This E-stability condition would be \(\alpha < 1\) if \(d\) increases to 1 from below. On the other hand, part (a) of the corollary says that the stability condition is \(\alpha \neq 1\) rather than \(\alpha < 1\) when \(d = 1\). The discontinuity arises from different assumptions on the knowledge possessed by economic agents. In the latter, agents are endowed with the knowledge of a stochastic trend in the data and estimate their model with differencing data.

The discontinuity disappears if we examine the stability condition under LDD when agents are endowed the knowledge or belief \(\rho = d\) as presented in part (b) of corollary 2.1. The stability condition there \(ad \neq 1\) will be \(\alpha \neq 1\) and identical to the stability condition in part (a) if \(d\) increases to 1 from below.

The stability condition \(\alpha \neq 1\) in part (a) of corollary 2.1 is robust and obtained when an alternative PLM is considered, i.e., \(\Delta y_t = a + \eta_t\) where agents learn about the (trend) growth of \(y_t\). The following result establishes the convergence of the learning about the trend growth process.

\(^{14}\)Although agents’ belief \(a_t\) converges to the RE value, this does not imply that they hold REE beliefs at the rest point. They still perceive incorrect persistence of residuals, i.e., \(\rho \neq 0\). Here I provide an example that the E-stability condition is identical to the condition governing the convergence of the adaptive learning process to a non-REE.
Proposition 2.2

Under all parameterizations $\alpha \neq 1$, agents’ trend growth belief $a_t$, i.e., when agents’ PLM is $\Delta y_t = a + \eta_t$, will converge to the RE value of the trend growth, i.e., zero. The ALM is $y_t = \frac{\delta}{1-\alpha} x_t + \frac{1}{1-\alpha} v_t$ and $y_t$ in equilibrium features larger volatility relative to RE if $0 < \alpha < 1$.

Proof. See Appendix B. \Halmos

2.2.4 Applications

Example 2.1 (Learning about the long-run growth of endogenous variables)

Most macroeconomic variables including market prices have positive growth rates in the long run, such as income, output, wages, stock prices, and goods prices. The literature on business cycle modeling with adaptive learning considers only learning about detrended variables. This way of specifying agents’ subjective beliefs implies that agents have exact knowledge about the trend growth of endogenous variables; see Kuang and Mitra (2014) for a detailed discussion.

In the following preliminary example, agents are assumed to be uncertain and learn about the (long-run) growth of endogenous variables, which requires estimating subjective models with differencing data. I study the convergence of the learning process and the associated convergence condition and discuss implications for empirical performance of the model with learning about long-run growth.

Take a simplified RBC model

$$\hat{k}_t = \alpha E_t \hat{k}_{t+1} + v_t$$

(10)

where $0 < \alpha < 1$. Denote by upper case letters the level of variables. $\hat{k}_t = \log K_t - \log P_t$ can be regarded as de-trended and log-linearized capital and

$$\log P_t = \log P_{t-1} + \xi_t$$

(11)

can be regarded as the logarithm of the productivity process. $v_t$ can be viewed as business cycle shocks independent of the permanent productivity shocks $\xi_t$. The RE solution is $\hat{k}_t = v_t$ in terms of detrended variables, or equivalently $\log K_t = \log P_t + v_t$ in terms of levels, or

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15 Alternatively, this could be interpreted as a simplified asset pricing model and $\hat{k}_t$ denotes detrended stock prices.

16 The steady state of $k_t$ is normalized to zero. It can be easily extended to the case that agents also learn about the steady state and that there is a nonzero drift term in the productivity process.
equivalently $\Delta \log K_t = \Delta \log P_t + \Delta v_t = \xi_t + \Delta v_t$ in terms of growth rates. The E-stability condition of the REE associated with the PLM $\hat{k}_t = a + \eta_t$ is $\alpha < 1$.

Reformulating the model equation (10) in levels yields

$$\log K_t = \alpha E_t \log K_{t+1} + (1 - \alpha) \log P_t + v_t$$

(12)

The model (11)-(12) corresponds to the model (1)-(2) with $y_t = \log K_t$, $\delta = 1 - \alpha$, $x_t = \log P_t$ and $d = 1$.

Now assume that agents are uncertain and learn about the trend growth, i.e., agents use the PLM $\Delta \log K_t = a + \eta_t$. The following results can be obtained according to proposition 2.2. (1) Agents’ long-run growth beliefs $a$ will converge to the RE value 0. (2) The ALM under LDD is $\log K_t = \log P_t + \frac{1}{1-\alpha} v_t$, which has the same functional form as the RE solution (i.e., $\log K_t = \log P_t + v_t$). Moreover, the detrended capital $\hat{k}_t$ under learning features excess volatility relative to RE because the business cycle shocks $v_t$ have a larger impact (i.e., $\frac{1}{1-\alpha}$) on the detrended capital under learning. (3) The rest point of the learning process is learnable if $\alpha \neq 1$.

The stability condition when agents learn about the trend (i.e., $\alpha \neq 1$) is looser than the E-stability condition when agents learn about detrended capital, i.e., $\alpha < 1$. The above result also illustrates that the precise knowledge with which agents are endowed by modelers, i.e., whether they know the trend growth or not, can affect the stability of the learning process in dynamic models. Of course in the two scenarios agents learn different objects: learning about the trend growth versus the steady state.

Kuang and Mitra (2014) obtain a similar result on the stability condition where agents learn about trend growth of endogenous variables in a stochastic growth model. We also find that shifting trend expectations is critical to understand business cycle fluctuations and has important implications for estimating output gap and cyclically-adjusted budget balance.

Now consider the alternative PLM $\Delta \log K_t = a \Delta \log P_t + \eta_t$ where agents regress differenced capital on differenced productivity. According to Part (a) of corollary 2.1, agents’ belief about $a$ will converge to the RE value 1. The ALM, stability condition and empirical performance under LDD associated with this alternative PLM is identical to the counterpart associated with the PLM with learning about the trend growth.

Many papers derive implications for monetary policy when agents are learning, such as Evans and Honkapohja (2003), Bullard and Mitra (2002), Gaspar, Smets and Vestin (2010). In these papers, agents learn about detrended endogenous variables and have exact knowledge about the trend growth. Taking into account agents’ uncertainty and learning

\footnote{For illustration, it is assumed here that the cyclical component of $\log K_t$ is zero or white noise.}
about the long-run growth will provide new insights in monetary policy design.

**Example 2.2** (Data (quasi-)differencing and excess volatilities of stock returns)

Consider a risk-neutral asset pricing model taking the form of model (1)-(2) with \(0 < \alpha < 1\). \(y_t\) and \(x_t\) represent stock prices and dividends respectively. \(\xi_t\) and \(\nu_t\) can be viewed as shocks to technology and subjective discount factor, respectively.

The RE solution is \(y_t = \frac{\delta}{1-\alpha d} x_t + \nu_t\) in terms of levels or equivalently \(\Delta y_t = \frac{\delta}{1-\alpha d} \Delta x_t + \Delta \nu_t\) in terms of differenced data. The ALM under learning with data (quasi-)differencing is equation (9) and hence \(y_t = \frac{\delta}{1-\alpha d} x_t + \frac{1}{1-\alpha \rho} \nu_t\) in levels or equivalently \(\Delta y_t = \frac{\delta}{1-\alpha d} \Delta x_t + \frac{1}{1-\alpha \rho} \Delta \nu_t\) in terms of (log) differenced data. Note \(x_t\) and \(\nu_t\) are independent, so do \(\Delta x_t\) and \(\Delta \nu_t\).

Based on proposition 2.1, the properties of the equilibrium under LDD are presented below. (1) Agents’ estimate of the parameter \(\alpha\) is unbiased relative to the corresponding RE value \(\frac{\delta}{1-\alpha d}\). (2) The equilibrium is associated with excess volatility of stock prices relative to the REE as the coefficient on \(\nu_t\) in the ALM is \(\frac{1}{1-\alpha \rho}\) which is larger than 1 if \(0 < \rho < \alpha^{-1}\). Stock returns under learning also display excess volatilities relative to RE as the coefficient on \(\Delta \nu_t\) is \(\frac{1}{1-\alpha \rho}\) in the ALM which is larger than 1 if \(0 < \rho < \alpha^{-1}\) and because the growth rate of stock prices is a major component of stock returns. Moreover, the volatility of stock prices and returns are increasing functions of agents’ perceived persistence of residuals \(\rho\). (3) The equilibrium is stable under learning if \(0 < \rho < \alpha^{-1}\).

A special case is when the dividend process follows a random walk process. In this case agents difference the data and estimate the relation between differenced stock prices and dividends. So we have \(d = \rho = 1\). The RE solution is \(y_t = \frac{\delta}{1-\alpha d} x_t + \nu_t\) in levels or \(\Delta y_t = \frac{\delta}{1-\alpha} \Delta x_t + \Delta \nu_t\) in terms of differenced variables. The resulting ALM under learning is \(y_t = \frac{\delta}{1-\alpha} x_t + \frac{1}{1-\alpha} \nu_t\) or \(\Delta y_t = \frac{\delta}{1-\alpha} \Delta x_t + \frac{1}{1-\alpha} \Delta \nu_t\). The growth rate of stock prices under learning features excess volatility relative to RE as the coefficient on \(\Delta \nu_t\) is \(\frac{1}{1-\alpha}\). Learning with data differencing strongly propagates the shocks \(\nu_t\) as \(\frac{1}{1-\alpha}\) can be arbitrarily large if the discount factor \(\alpha\) is close to 1.

### 2.3 Growth Rate Learning and Fragility of Rational Bubble Solutions

The literature has studied whether adaptive learning can rule out rational bubble solutions. For example, Evans (1989) studies the E-stability properties of bubbles solutions and find they are E-unstable in a range of RE models; see also Section 9.7 of Evans and Honkapohja (2001). More recently, McCallum (2009a, 2009b), Cochrane (2009, 2011) and Evans and
McGough (2010) discussed the stability of the bubble solution in the New Keynesian model, in particular when the underlying and exogenous driving process is not observable and cannot be taken as regressors.\footnote{Evans and McGough (2011) find that the bubble solutions in the one-equation, stripped down New Keynesian model are E-stable but not robustly E-stable when agents cannot observe the exogenous driving process or shocks.}

The literature performs a E-stability analysis associated with learning an explosive process. Noting the long-run growth rate of a bubbly variable (say stock prices) is constant in the RE bubble solution. I consider an alternative expectation formation method: learning about the growth rates of the bubbly variable.

To compare with Evans (1989), I set $d = 0$ and $\delta = 1$ in model (1)-(2). The model then coincides with the model (2) of Evans (1989) discussed in his Appendix A with the intercept $k = 0$. Assuming $0 < \alpha < 1$. Consider the bubble solutions

$$y_t = \alpha^{-1} y_{t-1} - \alpha^{-1} x_{t-1} + \epsilon_t$$

where $\epsilon_t$ is an arbitrary martingale difference sequence. Iterating the solution one period backward yields

$$y_t = \alpha^{-2} y_{t-2} - \alpha^{-2} x_{t-2} - \alpha^{-1} x_{t-1} + \epsilon_t + \alpha^{-1} \epsilon_{t-1}$$

Under RE the growth rate of the bubbly variable (say stock prices in the asset pricing model) is $\frac{y_t}{y_{t-2}} = \alpha^{-2}$ as $t$ tends to infinity or

$$\log y_t - \log y_{t-2} = -2 \log \alpha.$$

Suppose agents have the PLM

$$\log y_t - \log y_{t-2} = a + \eta_t \tag{13}$$

where agents perceive the variance of $\eta_t$ is small or vanishing. The PLM can be named as “growth rate learning” similar to “steady state learning” in the literature. Using data $y_{t-2}$ instead of $y_{t-1}$ is to avoid a singularity problem of the T-map. If one period is a quarter, then the PLM implies that agents use a semi-annualized forecasting model. Conditional expectations are $E_t y_{t+1} = e^a y_{t-1}$ given the perception that the variance of $\eta_t$ is vanishing. Substituting conditional expectations into the model equations yields the ALM under learning $y_t = \alpha e^a y_{t-1} + x_t + \nu_t$. The following proposition establishes the convergence of the learning process.

**Proposition 2.3**
Agents’ beliefs about the growth rate of the bubbly variable will converge to the RE value, i.e., \( a^* = -2 \log \alpha \). However, the rest point is unstable under learning with PLM (13).

**Proof.** See Appendix C. ■

The result obtained here when agents learn about the growth rate of the bubbly variable reinforces the results of Evans (1989), McCallum (2009a), and Evans and McGough (2011) arguing the fragility of the rational bubble solutions.\(^{19}\)

### 3 Models with Lagged and Stochastic Variables

Some different results arise when the model contains lagged and stochastic variables. Consider the model

\[
y_t = \alpha E_t y_{t+1} + \delta y_{t-1} + v_t
\]

(14)

Examples of models take this form are capital equation from a stochastic growth model or a New Keynesian Philips Curve with inflation indexation. \( v_t \) is assumed to be i.i.d process with zero mean.

#### 3.1 Regressors Contain Lagged Endogenous Variables

**3.1.1 REE and Stability under Least Squares Learning**

Assuming agents’ PLM under learning

\[
y_t = a y_{t-1} + w_t
\]

(15)

\[
w_t = \rho w_{t-1} + \eta_t
\]

(16)

They perceive \( \eta_t \) to be an i.i.d process. Assuming agents do not update their perceived persistence of residuals \( \rho \) in this section.

To calculate the REE, I set \( \rho = 0 \) and \( w_t = b v_t \) in equation (15)-(16). Agents’ conditional expectations are \( E_t y_{t+1} = a y_t \). Plugging the conditional expectations into the model (14), I get the ALM for \( y_t \)

\[
y_t = T_1^o(a) y_{t-1} + T_2^o(a) v_t
\]

(17)

where \( T_1^o(a) = \frac{\delta}{1-\alpha} \) and \( T_2^o(a) = \frac{1}{1-\alpha \delta} \). Two rational expectations solutions are \( a_{+,-} = \frac{1+\sqrt{1-4\alpha \delta}}{2\alpha} \) and \( b_{+,-} = \frac{1}{1-\alpha a_{+,-}} \), which can be found by solving \( T_1^o(a) = a \) and \( T_2^o(a) = b \). Below \( a^{RE}, b^{RE} \) are used to represent the coefficients in the REE of interest.

\(^{19}\)It can be shown that the instability of the explosive equilibrium can also be demonstrated by assuming agents use a similar PLM \( \frac{y_t}{y_{t-2}} = a + \eta_t \).
The stability condition for the REE under least squares learning is determined by the stability of the ODE (5) where $\phi = a$ and $T(\phi) = T^a_t(a)$, which can be shown to be $
frac{a\delta}{(1-aa+\ldots^2)} < 1$.

### 3.1.2 Learning with Data (Quasi-)Differencing (LDD)

Suppose agents hold beliefs (15)-(16) with $\rho \neq 0$. Estimating $a$ in (15) – (16) is equivalent to the following two-step procedure. Agents (quasi-)difference the regressand and regressors in the first step and in the second step estimate by OLS the following transformed model

$$y_t - \rho y_{t-1} = a(y_{t-1} - \rho y_{t-2}) + \eta_t$$

Similar to the learning algorithm (7)-(8), the recursive formulation of the estimation with data (quasi-)differencing is

$$a_t = a_{t-1} + \frac{1}{t}S_{t-1}^{-1}(y_{t-2} - \rho y_{t-3})[(y_{t-1} - \rho y_{t-2}) - (y_{t-2} - \rho y_{t-3})a_{t-1}]$$

$$S_t = S_{t-1} + \frac{1}{t}[(y_{t-1} - \rho y_{t-2})^2 - S_{t-1}]$$

Leading equation (18) for one period and taking conditional expectations deliver $E_t y_{t+1} = (a + \rho)y_t - a \rho y_{t-1}$. Plugging the conditional expectations into the model (14) delivers the ALM for $y_t$ under LDD

$$y_t = T_1(a; \rho)y_{t-1} + T_2(a; \rho)v_t$$

where $T_1(a; \rho) = \frac{\delta - a \rho}{1-\alpha(a+\rho)}$, and $T_2(a; \rho) = \frac{1}{1-\alpha(a+\rho)}$. For notational simplicity, I suppress $\rho$ in $T_1(a; \rho)$ and $T_2(a; \rho)$, so $T_1(a)$ and $T_2(a)$ are also used to represent them, respectively.

### 3.1.3 Convergence Results

Define $\theta = (a \; S)'$. Appendix D applies the Stochastic Recursive Algorithm elaborated in Evans and Honkapohja (2001) to analyze the convergence of the learning process. The stability of the learning model is determined by the stability of the associated ODE

$$\frac{d\theta}{dT} = H(\theta) = \begin{pmatrix} H_a(a, S) \\ H_S(a, S) \end{pmatrix}$$

where $H(\theta)$ are presented in Appendix D. The following proposition provides the convergence and stability of the LDD process.

**Proposition 3.1**
(a) When $\rho \neq 0$, $\rho \neq a^{RE}$, and the regressors contain lagged endogenous variables, agents’ belief $a_t$ under the LDD process will converge to rest point(s) determined by equation (65) instead of the RE value under the condition $\frac{\partial H_0(a^*,s^*)}{\partial a} < 0$.

(b) If $\rho > 0$, then $T_1(a^*;\rho) > a^*$. If $\rho < 0$, then $T_1(a^*;\rho) < a^*$.

(c) $\frac{\partial T_1}{\partial \rho}|_{a^* = 0, \rho = a^{RE}} = \frac{a^{RE}}{1 - a^{RE}}$.

**Proof.** See Appendix D. ■

Part (a) says that agents’ limiting belief under LDD is biased and derives the rest point(s) and the associated stability condition. Part (b) provides results on the direction of the bias. Part (c) provides a result on the derivative of the persistence of $y_t$ in the ALM under learning with respect to agents’ perceived persistence of residuals $\rho$ evaluated at the RE beliefs,\textsuperscript{20} which will be used in the numerical examples later.

An alternative derivation of the rest point(s) of the learning process is provided below to illustrate the intuition. Substituting the ALM (21) for $y_t$ and $y_{t-1}$ evaluated at $a = a^*$ into agents’ second step OLS regression with (quasi-)differenced data delivers

$$y_t - \rho y_{t-1} = T_1(a^*;\rho)(y_{t-1} - \rho y_{t-2}) + T_2(a^*;\rho)(v_t - \rho v_{t-1}) \quad (22)$$

The term $v_t - \rho v_{t-1}$ contains the disturbance $v_{t-1}$, which correlates with $y_{t-1}$ in the transformed regressors if $\rho \neq 0$. Applying OLS to (18) yields a biased limiting belief due to the correlation between the transformed regressors and residuals.

This result is analogous to a result in classical econometrics when the regressors contain lagged endogenous variables. If the residuals are serially correlated, the OLS estimator is inconsistent due to the correlation of the residuals and the regressors.\textsuperscript{21} Here under learning, if the true disturbances are not serially correlated and agents use OLS with data (quasi-)differencing when they detect serial correlation in the residuals, (quasi-)differencing the data with incorrectly perceived persistence of the residuals introduces correlation between the regressors and the residuals, so their limiting belief is biased.

Second, since the rest point(s) differ from the RE value, the stability condition of the rest point(s) under LDD will also differ from the E-stability condition of the REE under least squares learning, as can be seen more clearly from Appendix D.

I provide an intuitive and alternative way to derive the equation (65) which determines $a^*$ or $T_1(a^*)$ as a function of $\rho$. According to equation (22), regressing $y_t - \rho y_{t-1}$ on $y_{t-1} - \rho y_{t-2}$

\textsuperscript{20}Note the REE can be expressed as $\rho = 0$ and $a^* = a^{RE}$ or alternatively as $\rho = a^{RE}$ and $a^* = 0$.

\textsuperscript{21}See Griliches (1961) for a detailed discussion of the magnitude of the bias in this case.
by OLS delivers the coefficient on $y_{t-1} - \rho y_{t-2}$
\[
  a^*(\rho) = T_1(a^*; \rho) + \frac{\text{Cov}(y_{t-1} - \rho y_{t-2}, T_2(a^*; \rho)v_t - \rho T_2(a^*; \rho)v_{t-1})}{\text{Var}(y_{t-1} - \rho y_{t-2})}
  = T_1(a^*; \rho) - \frac{\rho(1 - T_2^2(a^*; \rho))}{(T_1(a^*; \rho) - \rho)^2 + (1 - T_2^2(a^*; \rho))}
\]  
(23)

From equation (23), it can be seen that positively (negatively) perceived persistence, i.e., $\rho > 0 (\rho < 0)$, introduces negative (positive) correlation between residuals and transformed regressors in the second step OLS regression with (quasi-)differenced data.\textsuperscript{22} So if $\rho$, agents’ perceived persistence of residuals, is larger (smaller) than its RE value 0, then agents’ estimate of the parameter in the PLM is biased downward (upward) relative to the parameter in the ALM as stated in part (b).

Part (b) is analogous to classical econometrics: OLS estimator is biased downward (upward) if the correlation between the residuals and the regressors is negative (positive). One difference here is that the parameter in the ALM at the rest point(s) depends on agents’ perceived persistence $\rho$.

A special case is that the residuals are perceived to contain a unit root, i.e., $\rho = 1$, which implies that the model of $y_t$ is perceived to be an ARIMA(1,1,0). In this case, agents will estimate their model with differencing data. Setting $\rho = 1$ in equation (23) yields $a^*(1) = \frac{T_1(a^*, 1)^{-1}}{2}$. Substituting the T-map delivers the following convergence result.

**Corollary 3.1**

If $\rho = 1$, agents’ beliefs $a_t$ will converge to $a^*(1) = \frac{(\delta + \alpha)^{-1}}{2}$ and the parameter in the ALM under learning $T_1(a; 1) = (\delta + \alpha)$ if the condition $\frac{\partial H_{a(a^*(1), S^*)}}{\partial a} < 0$ is satisfied.

### 3.1.4 Applications

In what follows two examples are provided to illustrate how data (quasi-)differencing matters for the empirical performance of learning models.

**Example 3.1** (Nearly self-fulfilling fluctuations: a RBC example)

An example of model class (14) is the log-linearized basic RBC model\textsuperscript{23} where $\alpha = 0.504, \delta = 0.494$ setting based on Giannitsarou (2007) and $y$ stands for the capital stock (in percentage deviation from the steady state) in this subsection. $v_t$ can be regarded as i.i.d productivity shocks.

\textsuperscript{22}I focus on stationary ALM so $|T_1(a^*, \rho)| < 1$.

\textsuperscript{23}For simplicity, Euler equation learning approach is followed in this and next example for illustration.
The REE is \( y_t = a^{RE} y_{t-1} + b^{RE} v_t = 0.958 y_{t-1} + 1.899 v_t \). Suppose under learning agents’ PLM takes the form\(^{24}\) (15)-(16). Given the beliefs, agents (quasi-)difference the capital data and estimate an AR(1) model to get an estimate of \( a \). Note \( a = 0 \) and \( \rho = 0.958 \) also corresponds to the REE.

The ALM under LDD is equation (21). Equation (23) determines agents’ limiting belief \( a^* \) and the persistence of capital in the ALM \( T_1(a^*, \rho) \) under learning. Figure 1 displays \( T_1(a^*, \rho) \) as a function of \( \rho \) over the interval \([0.8, 1]\) together with the capital persistence under RE.

According to part (a) and (b) of proposition 3.1, agents’ limiting belief \( a^* \) is biased downward relative to the actual capital persistence \( T_1(a^*, \rho) \). Part (c) of proposition 3.1 says that the derivative of actual capital persistence under learning (i.e., \( T_1(a^*, \rho) \)) with respect to agents’ perceived persistence \( \rho \) equals \( \frac{\alpha a^{RE}}{1 - \alpha a^{RE}} = 0.934 \) at \( \rho = a^{RE} = 0.958 \). As can be seen from figure 1, \( T_1(a^*, \rho) \) is a roughly linear and increasing function of \( \rho \) in the neighborhood of \( \rho = a^{RE} = 0.958 \) and over the whole interval. When agents’ perceived persistence \( \rho \) goes above (below) the RE value 0.958, the persistence and volatilities of capital increase (decrease). Moreover, agents’ beliefs are nearly self-fulfilling under learning with data (quasi-)differencing because the actual persistence of capital is very close to agents’ perceived persistence of capital. Numerical calculations confirm that the equilibria under LDD associated with figure 1 are learnable for all \( \rho \) in the interval \([0.8, 1]\).

A special case is \( \rho = 1 \) where agents perceive a stochastic trend. In this case the ALM under LDD is \( y_t = 0.997 y_{t-1} + 1.974 v_t \). The ALM for \( y_t \) is a very persistent process

\(^{24}\) Agents perceive that the residuals are serially correlated or that they miss some unobserved persistent driving process (including a random walk process of \( w_t \) as a special case).
Figure 2: Inflation persistence in the ALM as a function of perceived persistence of residuals and features excess volatilities relative to RE. LDD helps a lot to propagate the business cycle shocks. The ALM under learning is very close to the PLM in this case, i.e., $y_t = 0.9986y_{t-1} + 0.0014y_{t-2} + \eta_t$. It is very difficult to distinguish the two processes with limited data statistically.\footnote{One way to see this is to plot the spectral densities of the two processes, which can be shown to match very well except at very low frequency.}

**Example 3.2** (Nearly self-fulfilling fluctuations: a New Keynesian example)

Another example of the model class (14) is New Keynesian Philips curve model with inflation indexation where $y$ stands for inflation deviation from a target. I set $\alpha = 0.66$, $\delta = 0.33$ according to Gaspar, Smets, and Vestin (2010).\footnote{This is obtained in the case when the discount factor is 0.99 in the utility function and the inflation indexation parameter is 0.5.} The innovations $\nu_t$ are assumed to be i.i.d for simplicity.

Assume agents’ PLM takes the form (15)-(16). So agents learn about inflation persistence with (quasi-)differencing data. The REE is $y_t = 0.50y_{t-1} + 1.4950\nu_t$ or equivalently $a = a^{RE} = 0$, $\rho = 0.50$ and $\eta_t = 1.4950\nu_t$.

The ALM under LDD is equation (21). Figure 2 plots the inflation persistence under LDD as a function of $\rho$ over the interval $[0.4, 1]$. For example, for the special case where agents perceive $\rho = 1$, the ALM under LDD is $y_t = 0.9902y_{t-1} + 2.9322\nu_t$, which is learnable indicated by numerical calculations. The PLM is $y_t = 0.9951y_{t-1} + 0.0049y_{t-2} + \eta_t$. The two processes are close to each other. The inflation persistence is very close to agents’ perception over the whole interval, so their beliefs are nearly self-fulfilling. The inflation persistence and
volatilities are increasing in $\rho$. Moreover, the persistence and volatilities are higher (lower) than under RE if $\rho$ is higher (lower) than the corresponding RE value.

The two examples suggest that data (quasi-) differencing may be an important source of business cycle fluctuations. The results relate to Sargent (1999), Branch and Evans (2011, 2013) which show random walk beliefs are nearly self-fulfilling in the model of Bray (1982), an asset pricing model and an New Keynesian model of inflation, respectively.\textsuperscript{27} It is shown here that not only the special case of random walk beliefs, but more generally beliefs about persistent residuals like in figure 1 or 2 are nearly self-fulfilling under LDD.\textsuperscript{28}

### 3.2 Self-fulfilling Data (Quasi-)Differencing

This section provides a case where data (quasi-) differencing is exactly self-fulfilling. Taking the model with lagged endogenous variables in Section 3.1 for an example. At the rest point $a^*$, the residuals in agents’ regression are

$$w_t = (T_1(a^*; \rho) - a^*)y_{t-1} + T_2(a^*; \rho)v_t$$  \hspace{1cm} (24)

Note $a^*$ is biased relative to the RE value and not a fixed point of the $T_1$ map, so $T_1(a^*; \rho) - a^*$ is non-zero.

Denote $\rho_w(\rho)$ the autocorrelation coefficient of residuals $w_t$ in the ALM under LDD. If $\rho_w(\rho^*) = \rho^*$, agents will find that the autocorrelation coefficient of the residuals $w_t$ equals their perceived persistence of residuals. Appendix E shows that $\rho_w(\rho^*) = \rho^*$ is equivalent to

$$(T_1(a^*; \rho^*) - a^*)(T_1(a^*; \rho^*) - \rho^*)(1 - T_1(a^*; \rho^*)^2)(T_1(a^*; \rho^*) - a^* - \rho^*) = 0$$  \hspace{1cm} (25)

Denote $(\Delta)$ the system of equations (23) and (25) which jointly determines $a^*$ and $\rho^*$. I focus on the case when $\rho \neq 0$ and $\rho \neq a^{RE}$ because the RE solution expressed by $a^* = a^{RE}$ and $\rho^* = 0$, and alternatively by $a^* = 0$ and $\rho^* = a^{RE}$ are solutions to $(\Delta)$. The following proposition summarizes the above discussion.

**Proposition 3.2**

\[ \exists \rho^* \neq 0 \text{ and } \rho^* \neq a^{RE} \text{ determined by the system (}\Delta\text{) such that the rest points of the LDD process have the property that agents’ wrong belief about the persistence of residuals } \rho^* \text{ is self-fulfilling, i.e., if agents calculate the serial correlation of the residuals in the data, it will equal their perceived persistence of the residuals } \rho^*. \]

\textsuperscript{27}Bullard, Evans and Honkapohja (2008) show that judgment in policy making could be nearly self-fulfilling in dynamic models.

\textsuperscript{28}Note also the model here has lagged endogenous variables not present in their models.
Proof. See Appendix E. ■

In the presence of biased limiting beliefs, serially correlated regressors will enter the residuals, which can generate serially correlated residuals and confirm agents’ wrong beliefs about the persistence of the residuals.

When agents perceive or detect serially correlated residuals, GLS estimation is recommended by econometricians. The use of GLS estimation can be exactly self-fulfilling that the serial correlation of residuals under self-referential learning is identical to agents’ subjective perception. This means that a simple check, i.e., calculating the serial correlation of residuals, cannot detect that they get the wrong estimates relative to the RE value. Either an alternative specification of the model or more careful diagnosis is needed to detect the problem in order to learn the REE.

4  Feasible Learning with Data (Quasi-)Differencing (FLDD)

I now turn to the case when agents also learn about the persistence of residuals $\rho$ over time. The learning algorithms are called feasible learning with data (quasi-)differencing and defined as follows.

**Definition 4.1** Feasible learning with data (quasi-)differencing (FLDD)

Agents learn about the persistence of residuals $\rho$ in their regression model over time and then apply learning with data (quasi-)differencing with estimated $\rho$.

4.1  Convergence of Learning Under-parameterized Models to REE

Serially correlated residuals arise when agents use misspecified and underparameterized models omitting some serially correlated (and possibly unobserved) variables. Least squares learning of an underparameterized models usually converges to a restricted perception equilibrium featuring serially correlated residuals, see Adam (2007) and Evans and Ramey (2006) for examples.

One way to address the misspecification is to use more sophisticated econometric methods than least squares estimator as econometricians do. This section provides a new perspective on the stability of a REE by establishing the convergence of learning underparameterized models to a REE. The condition governing the convergence is called “Strong E-stability condition II” and defined below. It also suggests analyzing the convergence of misspecified models has important implications for addressing model misspecifications in econometric practice.
Definition 4.2 Strong E-stability condition II

The strong E-stability condition II is defined as the stability condition of the REE when agents use FLDD (i.e., a slightly more sophisticated econometric method than ordinary least squares) to learn underparameterized models.

The name is similar to strong E-stability condition governing the convergence of learning over-parameterized models in the literature. “II” is appended in the name because a different type of misspecification (i.e., underparameterization) is considered rather than over-parameterization.

Consider again the model (14) with lagged endogenous variables in Section 3. Agents are assumed to use the following underparameterized model

\[ y_t = a + \omega_t \] (26)

omitting lagged variables \( y_{t-1} \) as regressors. It can be shown that least squares learning of equation (26) will yield a limiting belief \( 0 \) for \( a \) and the residuals are serially correlated with correlation coefficient \( \delta \). The resulting equilibrium is often called a “restricted perception equilibrium.”

Suppose agents are alert to possible model misspecifications, allowing for a richer structure of the variance-covariance matrix. Specifically, agents perceive serially correlated residuals \( \omega_t = \rho \omega_{t-1} + \eta_t \). They learn the persistence of residuals and use FLDD to learn about \( a \) in (26).

(Quasi-)Differencing equations (26) with \( \rho \) yields

\[ y_t = a(1 - \rho) + \rho y_{t-1} + \eta_t \] (27)

Note (27) is a restricted version of

\[ y_t = c + j y_{t-1} + \epsilon_t \] (28)

First, (28) is estimated by OLS, delivering \( j \) as a consistent estimate of \( \rho \). Second, agents apply OLS to equation (27) with the quasi-differenced data with estimated \( \rho \).

Agents’ conditional expectations are \( E_t y_{t+1} = a(1 - j) + j y_t \). The ALM under FLDD is

\[ y_t = T_1(a) + T_2(a) y_{t-1} + T_3(a) \nu_t \] (29)

where \( T_1(a) = \frac{a}{1-\alpha_j}, T_2(a) = \frac{\delta}{1-\alpha_j}, T_3(a) = \frac{1}{1-\alpha_j} \).
The FLDD algorithm is

\[
a_t = a_{t-1} + \frac{1}{t} S^G_{t-1} (1 - j_t - 2) (y_{t-1} - j_t - y_{t-2} - a_{t-1} (1 - j_t))
\]

\[
S^G_t = S^G_{t-1} + \frac{1}{t} ((1 - j_t - 2)^2 - S^G_t)
\]

\[
 \begin{pmatrix} c_t \\ j_t \end{pmatrix} = \begin{pmatrix} c_{t-1} \\ j_{t-1} \end{pmatrix} + \frac{1}{t} S^O_{t-1} \left( \begin{pmatrix} 1 \\ y_{t-2} \end{pmatrix} \right) (y_{t-1} - c_{t-1} - j_{t-1} y_{t-2})
\]

\[
S^O_t = S^O_{t-1} + \frac{1}{t} \left( \begin{pmatrix} 1 \\ y_{t-1} \end{pmatrix} \right) (1 - y_{t-1} - S^O_{t-1})
\]

Note the REE, i.e., \( a = 0 \) and \( \rho = \rho^{RE} \), is the rest point of the learning process. The following proposition summarizes the convergence result.

**Proposition 4.1**

*Feasible Learning with data (quasi-)differencing* of the under-parameterized model (26) will converge to the REE of the model if \( \frac{a}{1 - \alpha a^{RE}} < 1 \), the E-stability condition \( \frac{\alpha a^{RE}}{1 - \alpha a^{RE}} < 1 \), and \( \left| \frac{\delta}{1 - a^{RE}} \right| < 1 \) are satisfied.

**Proof.** See Appendix F. □

Convergence of learning the underparameterized model to the REE here is intuitive by noting that the REE in this model can be alternatively expressed as an underparameterized model plus serially correlated residuals. Appendix F shows that the convergence condition can be derived by the *E-stability principle*, i.e., the stability of the ODE (5) where \( \phi = (a, c, j)' \) and \( T(\phi) = \left( \frac{T_1(\phi)}{T_2(\phi)}, T_1(\phi), T_2(\phi) \right)' \) and \( T_1- \) and \( T_2- \) map are defined just after equation (29). This gives \( \frac{\alpha}{1 - \alpha a^{RE}} < 1 \) and \( \frac{a^{RE}}{1 - \alpha a^{RE}} < 1 \). Stationarity of the ALM under learning requires \( \left| \frac{\delta}{1 - \alpha a^{RE}} \right| < 1 \).

Note the *Strong E-stability condition II* can be tighter than the E-stability condition of the REE under least squares learning. For example, if \( 0 < a^{RE} < 1 \), then \( \frac{\alpha}{1 - \alpha a^{RE}} < 1 \) can be tighter than the E-stability condition \( \frac{\alpha}{1 - \alpha a^{RE}} < \frac{1}{a^{RE}} \).

When we start with a misspecified and underparameterized model, there are different ways to address the misspecification in order to learn the REE. As in the example, the condition governing the convergence of learning misspecified models to the REE with more sophisticated methods than OLS (i.e., the *strong E-stability condition II*) is tighter than the condition governing the convergence of least squares learning the adjusted and correctly specified model to the REE (i.e., the E-stability condition of the REE under least squares learning). This implies that in the example adjusting the model specification and learn
using OLS are more desirable than adopting more sophisticated estimation methods because
instability of the REE or divergence of the estimates happens more likely in the latter.

The literature studies the convergence of learning over-parameterized models to a REE
and derives the associated stability condition of the REE. The condition is called Strong E-

stability condition. If the strong E-stability condition is tighter than the E-stability condition
of the REE under least squares learning, using over-parameterized model is less desirable
than adjusting the model specification because the former more likely leads to the divergence
of the REE.

Generally, comparing the condition governing the convergence of learning misspecified
(either over-parameterized or under-parameterized) models with the E-stability condition
under least squares learning is informative and can be used as a criteria for evaluating the
desirability of different ways to address model misspecifications in econometric practice.

The next sections provide the recursive formulation of several FLDD algorithms deliv-
ering consistent estimates of $\rho$, establish the convergence of FLDD of correctly-specified
models to the REE and derive the associated stability condition. A numerical example sug-
gests that learning $\rho$ over time and estimating models with (quasi-)differencing data can be
used to explain endogenous drifting and regime switching in persistence and volatilities of
macroeconomic activities.

### 4.2 Model with Both Exogenous and Lagged Endogenous Variables

Consider the following model with both exogenous and lagged endogenous variables

\[
\begin{align*}
y_t & = \alpha E_t y_{t+1} + \delta y_{t-1} + \gamma x_t \\
x_t & = dx_{t-1} + A_t
\end{align*}
\]

where $A_t$ is an i.i.d. innovation, $d \neq 0$ and $|d| < 1$.

Suppose agents perceive the following model

\[
\begin{align*}
y_t & = ay_{t-1} + bx_{t-1} + \eta_t \\
\eta_t & = \rho \eta_{t-1} + \omega_t
\end{align*}
\]

Equation (32)-(33) can be transformed to

\[
y_t = (a + \rho)y_{t-1} - a y_{t-2} + bx_{t-1} - b \rho x_{t-2} + \omega_t
\]

\[\text{29}\text{See Chapter 9 of Evans and Honkapohja (2001) for an example.}\]
Agents’ conditional expectations are \( E_t y_{t+1}^G = (a + \rho)y_t - ax_{t-1} + bx_t - bx_{t-1} \). Substituting the expectations into model (30) yields the ALM

\[
y_t = T_1(a, b)y_{t-1} + T_2(a, b)x_{t-1} + T_3(a, b)A_t
\]

(35)

where \( T_1(a, b) = \frac{\delta - \alpha \omega \rho}{\delta - \alpha (\alpha + \rho)} \), \( T_2(a, b) = \frac{\alpha b + \gamma d}{\delta - \alpha (\alpha + \rho)} \), and \( T_3(a, b) = \frac{\alpha d + \gamma c}{\delta - \alpha (\alpha + \rho)} \).

The FLDD algorithm augments the LDD algorithm by estimating \( \rho \) consistently via a procedure of Durbin (1960). First, agents estimate

\[
y_t = cy_{t-1} + ey_{t-2} + f x_{t-1} + g x_{t-2} + \omega_t
\]

(36)

by OLS. Note equation (34) is a restricted version of equation (36). Second, agents get the consistent estimate \( \rho = -\frac{\delta}{\delta - \alpha} \).

Define \( \tilde{a}_t = \begin{pmatrix} c_t & e_t & f_t & g_t \end{pmatrix}' \) and \( \tilde{y}_{t-1} = \begin{pmatrix} y_{t-1} & y_{t-2} & x_{t-1} & x_{t-2} \end{pmatrix}' \). The FLDD algorithm is

\[
\begin{pmatrix} a_t \\ b_t \\ S_t^G \\ \tilde{a}_t \\ S_t^O \\ \rho_t \end{pmatrix} = \begin{pmatrix} a_{t-1} \\ b_{t-1} \\ S_{t-1}^G \\ \tilde{a}_{t-1} \\ S_{t-1}^O \\ \frac{g_t}{f_t} \end{pmatrix} + \frac{1}{t} \begin{pmatrix} y_{t-2} - \rho_{t-2} y_{t-3} \\ x_{t-2} - \rho_{t-2} x_{t-3} \\ y_{t-1} - \rho_{t-2} y_{t-2} \\ x_{t-1} - \rho_{t-2} x_{t-3} \\ x_{t-1} - \rho_{t-1} x_{t-2} - S_{t-1}^G \end{pmatrix}
\]

(37)

\[
S_t^G = S_{t-1}^G + \frac{1}{t} \begin{pmatrix} y_{t-1} - \rho_{t-1} y_{t-2} \\ x_{t-1} - \rho_{t-1} x_{t-2} \end{pmatrix} \begin{pmatrix} y_{t-1} - \rho_{t-1} y_{t-2} \\ x_{t-1} - \rho_{t-1} x_{t-2} \end{pmatrix} - S_{t-1}^G
\]

(38)

\[
\tilde{a}_t = \tilde{a}_{t-1} + \frac{1}{t} S_{t-1}^O \begin{pmatrix} y_{t-2} - c_{t-1} y_{t-2} - e_{t-1} y_{t-3} - f_{t-1} x_{t-2} - g_{t-1} x_{t-3} \end{pmatrix}
\]

(39)

\[
S_t^O = S_{t-1}^O + \frac{1}{t} \begin{pmatrix} \tilde{y}_{t-1} - \tilde{y}_{t-1} \end{pmatrix} - S_{t-1}^O
\]

(40)

\[
\rho_t = -\frac{g_t}{f_t}
\]

(41)

(39)-(41) are used to obtain a consistent estimate of the persistence of residuals. The following result summarizes the convergence of the learning process.

**Proposition 4.2**

If the E-stability condition of the REE under least squares learning holds, then the recursive learning process (37)-(41) will converge to the REE.

**Proof.** See Appendix G. ■

Note in this procedure a model with more lags, i.e., equation (36) needs to be estimated to get a consistent estimate of \( \rho \). The consistent estimation procedure ensures the convergence of \( \rho \) to the RE value, so that agents’ estimate of the parameters will converge to the RE values.
Appendix G shows that the condition governing the convergence of the FLDD process is determined by the E-stability Principle or the stability of the ODE (5) where \( \phi = (a, b, c, e, f, g)' \) and \( T(\phi) = (T_1(\phi), T_2(\phi), T_1(\phi), 0, T_2(\phi), 0)' \) and \( T_1- \) and \( T_2- \) map are defined just after equation (35). Also note the convergence condition is identical to the E-stability condition of the REE under least squares learning in this case.

### 4.3 Regressors Contain Only Lagged Endogenous Variables

I return to the model with lagged endogenous variables as regressors in section 3.1. The procedure described in section 4.2 to learn the persistence of residuals does not work due to lack of exogenous variables as regressors. This subsection proposes a procedure to estimate the persistence of residuals \( \rho \) in this case and studies the convergence of the FLDD process.

Note agents’ PLM (15) – (16) is a restricted version of the following equation

\[
y_t = cy_{t-1} + my_{t-2} + \eta_t
\]

with \( c = a + \rho \) and \( m = -a\rho \). The following procedure is considered. First, agents estimate (42) by OLS. I focus on the scenario \( c^2 + 4m > 0 \) which ensures that the eigenvalues associated with the difference equation (42) are within the unit circle.\(^3\) Agents obtain \( \rho = \frac{c + \sqrt{c^2 + 4m}}{2} \) if \( c > 0 \) and \( \rho = \frac{c - \sqrt{c^2 + 4m}}{2} \) if \( c < 0 \). This leads to a consistent estimate of \( \rho \). Second, agents employ OLS to the quasi-differenced data with estimated \( \rho \).

The recursive FLDD algorithm is as follows

\[
\begin{align*}
a^G_t &= a^G_{t-1} + \frac{1}{t}((S^G_{t-1})^{-1}(y_{t-3} - \rho_{t-2}y_{t-4})(y_{t-2} - \rho_{t-2}y_{t-3} - (y_{t-3} - \rho_{t-2}y_{t-4})a^G_{t-1}) \quad (43) \\
S^G_t &= S^G_{t-1} + \frac{1}{t}((y_{t-2} - \rho_{t-1}y_{t-3})^2 - S^G_{t-1}) \quad (44) \\
\begin{pmatrix} c_t \\ m_t \end{pmatrix} &= \begin{pmatrix} c_{t-1} \\ m_{t-1} \end{pmatrix} + \frac{1}{t}(S^O_{t-1})^{-1} \begin{pmatrix} y_{t-3} \\ y_{t-4} \end{pmatrix} (y_{t-2} - c_{t-2}y_{t-3} - m_{t-2}y_{t-4}) \quad (45) \\
S^O_t &= S^O_{t-1} + \frac{1}{t} \left( \begin{pmatrix} y_{t-2} & y_{t-3} \end{pmatrix}' \begin{pmatrix} y_{t-2} & y_{t-3} \end{pmatrix} - S^O_{t-1} \right) \quad (46) \\
\rho_t &= \begin{cases} 
\frac{c_t + \sqrt{c_t^2 + 4m_t}}{2} & \text{if } c_t \geq 0 \\
\frac{c_t - \sqrt{c_t^2 + 4m_t}}{2} & \text{if } c_t < 0
\end{cases} \quad (47)
\end{align*}
\]

The following proposition summarizes the convergence result.

**Proposition 4.3**

\(^3\)If this inequality is not satisfied, a projection facility will be needed. Standard argument can then be applied.
The recursive learning process (43) – (47) will converge to the REE if the E-stability condition of the REE under least squares learning is satisfied.

**Proof.** See Appendix H.

Appendix H shows that the E-stability principle still holds so the convergence condition is determined by the stability of the ODE (5) where \( \phi = (a^G, c, m)^T \) and \( T(\phi) = (T_1(\phi), T_1(\phi), 0)' \) and \( T_1 - \) map is defined just after equation (21). The convergence condition is identical to the E-stability condition of the REE under least squares learning.

### 4.4 An Application

**Example 4.1** *(Example 3.1 continued)*

Example 3.1 is reconsidered here. Nevertheless, agents learn about the persistence of the residuals in their regression. The FLDD procedure (43) – (47) is employed except that a small constant gain sequence instead of the decreasing gain sequence \( \frac{1}{T} \) is utilized. Figure 3 presents a representative simulation with constant gain parameter 0.03. The upper panel displays the evolution of the endogenous variable \( y_t \). The middle panel presents agents’ perceived persistence \( \rho \) and the lower panel displays the unconditional variance of \( y_t \) calculated as moving averages with window length 150 of the variance of the simulated time series.\(^{31}\)

Recall the PLM takes the form (15)-(16). The RE solution is \( a = 0 \) and \( \rho = 0.958 \). Agents’ perceived persistence of residuals fluctuates over time and can be higher or lower than the RE value 0.958. As can be seen from the lower panel, the volatilities of \( y_t \) switch between low and high regime. When agents’ perceived persistence of residuals is higher than the RE value 0.958 or close to 1, the persistence and volatilities of \( y_t \) tend to be high. Similarly, when agents’ perceived persistence of residuals is low relative to the RE value, the resulting persistence and volatilities of \( y_t \) tend to be low.

Learning with estimating perceived residual persistence and data filtering is one channel and can be used to endogenously generate drifting and regime switching in the volatilities of inflation and output volatilities in the US during the postwar as documented in Cogley and Sargent (2005) and Sims and Zha (2006). Alternative channels generating this phenomenon are the combination of learning and dynamic predictor selection as in Branch and Evans (2007). My model shares some similarity with their model. In my paper, agents look backward the forecast errors, estimate the persistence of the forecast errors and choose the estimation method or the degree of data filtering. In Branch and Evans (2007), agents look

\(^{31}\) Projection facility is imposed, so agents’ perceived persistence of residuals will not be updated but set to the belief at last period if \( c_t^2 + 4 m_t < 0 \) or \( \rho_t > 1 \).
4.5 Current Period Exogenous and Stochastic as Regressors

I return to the model with exogenous variables (1)-(2) considered in section 2 but assume agents learn about the persistence of residuals $\rho$ over time here. The ALM for $y_t$ under learning is still equation (9) with $\rho$ replaced by a consistent estimate $\rho_t$. The following consistent feasible GLS procedure is recommended in classical econometrics (say by Greene (2003)) to estimate $\rho$ when the regressors contain only exogenous variables. Firstly, agents run an OLS regression and compute regression residuals. Then they estimate $\rho$ by running an OLS regression using the residuals. The persistence coefficient is estimated as follows

$$
\rho_t = \frac{\sum_{i=1}^{t-2} w_i w_{i+1}}{\sum_{i=1}^{t-2} w_i^2} = \frac{\sum_{i=1}^{t-2} (y_i - x_i a_t^O)(y_{i+1} - x_{i+1} a_t^O)}{\sum_{i=1}^{t-2} (y_i - x_i a_t^O)^2}
$$

where $\hat{w}_t$ is the residual from the OLS estimation and $a_t^O$ the OLS estimator. Second, agents apply OLS to the quasi-differenced data with estimated $\rho$. 

Figure 3: Evolution of $y_t$, agents’ perceived persistence of residuals and variance of $y_t$ under learning
The FLDD algorithm is

\[
\begin{align*}
    a_t^G &= a_{t-1}^G + \frac{1}{t} (S_{t-1}^G)^{-1} (x_{t-2} - \rho_{t-2} x_{t-3})(y_{t-2} - \rho_{t-2} y_{t-3} - (x_{t-2} - \rho_{t-2} x_{t-3})a_{t-1}^G) \\
    S_t^G &= S_{t-1}^G + \frac{1}{t} ((x_{t-1} - \rho_{t-1} x_{t-2})^2 - S_{t-1}^G) \\
    a_t^O &= a_{t-1}^O + \frac{1}{t} (S_{t-1}^O)^{-1} x_{t-1}(y_{t-1} - a_{t-1}^O x_{t-1}) \\
    S_t^O &= S_{t-1}^O + \frac{1}{t} (x_{t}^2 - S_{t-1}^O) \\
    \rho_t &= \rho_{t-1} + \frac{1}{t} (S_{\rho,t-1})^{-1} (y_{t-2} - a_{t-2}^O x_{t-2})(y_{t-1} - a_{t-1}^O x_{t-1} - (y_{t-2} - a_{t-2}^O x_{t-2})\rho_{t-1}) \\
    S_{\rho,t} &= S_{\rho,t-1} + \frac{1}{t} ((y_{t-1} - a_{t-1}^O x_{t-1})^2 - S_{\rho,t-1})
\end{align*}
\] (48-53)

where \(a_t^G\) is the estimator with (quasi-)differencing data. (50)-(53) deliver a consistent estimate of \(\rho\).

Two differences of the learning algorithm (48)-(53) from the above described procedure are as follows. First, in the second step of the procedure, current estimate about the serial correlation is used to quasi-difference both the latest and past data. But in equation (48), only latest data are quasi-differenced with \(t_2\), while past data are quasi-differenced with past perceived persistence of the residuals. Second, it is known that the appearance of current belief on the right hand side of the learning algorithm will bring a more complicated fixed-point problem so I use \(\rho_{t-2}\) in (48) and hence \(\rho_{t-1}\) will appear in (49).

**Proposition 4.4**

The recursive learning process (48)-(53) will converge to the REE if the E-stability condition of the REE under least squares learning is satisfied.

**Proof.** See Appendix I. ■

Recall Proposition 2.1 that when agents do not learn about \(\rho\), their estimate about \(a^G\) will converge to the RE value for any \(\rho\). Once agents’ belief \(a^G\) equals the RE value, they will find out that the serial correlation of the residuals is zero and learn to form RE.

## 5 On Equilibrium Selection

Econometricians/agents very often adopt more sophisticated estimation methods (e.g., FLDD) than ordinary least squares, being alert to possible model misspecifications or allowing for a richer variance-covariance structure of residuals. This implies that the specification of agents’ econometric model is more general and flexible such that it can nest RE equilibria...
with different functional forms in models with multiple RE equilibria. An associated issue is: which REE will the FLDD process converge to and at which condition? To address this question, this section considers a class of models with multiple equilibria, an example of which is an overlapping generations model with fiat money.\footnote{Different versions of the model have been considered by Duffy (1994), Woodford (1990), Adam (2003), among others.}

### 5.1 Model and RE Equilibria

The model is

\[
y_t = \alpha + \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + v_t
\]

where \( v_t \) is an exogenous process satisfying \( E_{t-1} v_t = 0 \). The derivation of a deterministic version can be found in Duffy (1994) where \( y_t \) is inflation. It is also presented in Chapter 8 and 9 of Evans and Honkapohja (2001). Other model examples taking such form are Sargent and Wallace (1975) and the real balance model of Taylor (1977).

The model has a MSV RE solution \( y_t = (1 - \beta_0 - \beta_1)^{-1} \alpha + v_t \) and also non-MSV RE solutions, such as an AR(1) solution \( y_t = -\beta_0^{-1} \alpha + \beta_1^{-1} (1 - \beta_0) y_{t-1} + v_t \).

### 5.2 Equilibrium Selection

Suppose agents perceive the following model of \( y_t \)

\[
\begin{align*}
y_t & = a^G + \omega_t \\
\omega_t & = \rho \omega_{t-1} + \eta_t
\end{align*}
\]

A consistent estimate of \( \rho \) is the coefficient on \( y_{t-1} \) when OLS is applied to estimate equation \( y_t = n + \rho y_{t-1} + \eta_t \). The ALM for \( y_t \) under FLDD is

\[
y_t = T_1(a^G_{t-1}; \rho_{t-1}) + T_2(a^G_{t-1}; \rho_{t-1}) y_{t-1} + v_t
\]

where \( T_1(a^G; \rho) = \alpha + \beta_0 (a^G (1 - \rho)) + \beta_1 (a^G (1 - \rho^2)) \) and \( T_2(a^G; \rho) = \beta_0 \rho + \beta_1 \rho^2 \).
The FLDD algorithm is:

\[
a_t^G = a_{t-1}^G + \frac{1}{t} S_{t-1}^G (1 - \rho_{t-1}) (y_{t-1} - \rho_{t-1} y_{t-2} - a_{t-1}^G (1 - \rho_{t-1})) \tag{58}
\]

\[
S_t^G = S_{t-1}^G + \frac{1}{t} ((1 - \rho_{t-1})^2 - S_{t-1}^G) \tag{59}
\]

\[
\begin{pmatrix} n_t \\ \rho_t \end{pmatrix} = \begin{pmatrix} n_{t-1} \\ \rho_{t-1} \end{pmatrix} + \frac{1}{t} S_{t-1}^G \left( \frac{1}{y_{t-1} - n_{t-1} - \rho_{t-1} y_{t-2}} \right) \tag{60}
\]

\[
S_t^O = S_{t-1}^O + \frac{1}{t} ((1 y_{t-1})' (1 y_{t-1}) - S_{t-1}^O) \tag{61}
\]

The following proposition summarizes the convergence result.

**Proposition 5.1**

(a) The learning with data (quasi-)differencing process has two rest points: the MSV solution (i.e., \(a^G = (1 - \beta_0 - \beta_1)^{-1} \alpha \) and \( \rho = 0 \)) and the AR(1) solution (i.e., \(a^G = \frac{-\beta_0 \alpha}{1 - \rho} \) and \( \rho = \beta_1^{-1} (1 - \beta_0) \)).

(b) If \( \beta_0 + \beta_1 < 1 \) and \( \beta_0 < 1 \), then the learning process (58)-(61) will converge to the MSV solution.

(c) If \( \beta_1 < 0, \beta_0 > 1 \) and \( |\frac{1 - \beta_0}{\beta_1}| < 1 \), then the learning process (58)-(61) will converge to the AR(1) solution.

**Proof.** See Appendix J. ■

Appendix J shows that the condition governing the convergence of the FLDD to the MSV solution is determined by the *E-stability principle*, i.e., the stabillity of the ODE (5) where \( \phi = (a^G, n, \rho)' \), \( T (\phi) = (T_1 (\phi), T_1 (\phi), T_2 (\phi))' \), and \( T_1 - \) and \( T_2 - \) map are defined just after equation (57). This yields the conditions in part (b) of the proposition.

Note the E-stability condition\(^{33}\) of the MSV equilibrium under least squares learning is not sufficient for ensuring the convergence of the FLDD process to the equilibrium. The latter coincides with the strong E-stability condition for the MSV equilibrium\(^{34}\) under least squares learning when agents’ PLM is over-parameterized with one more lag. This is because the FLDD requires estimation of agents’ model with one more lag of variables. Learning about the persistence of residuals imposes further restriction on the parameters to ensure the convergence.

Part (c) of the proposition says that under certain conditions the learning process will converge to the AR(1) solution. Appendix J shows that the convergence condition can be

\(^{33}\)See page 180 of Evans and Honkapohja (2001) for the E-stability condition of the MSV equilibrium under least squares learning, which is \( \beta_0 + \beta_1 < 1 \).

\(^{34}\)See page 188 of Evans and Honkapohja (2001) the strong E-stability condition of the MSV equilibrium under least squares learning for this model, which is \( \beta_0 + \beta_1 < 1 \) and \( \beta_0 < 1 \).
obtained via the *E-stability principle* and determined by the stability of the ODE (5) where
\[ \phi = (a^G, n, \rho)' \]
\[ T(\phi) = \left( \frac{T_1(\phi)}{1-T_2(\phi)}, T_1(\phi), T_2(\phi) \right)' \]
and \( T_1 - \) and \( T_2 - \) map are defined just after equation (57). This yields \( \beta_1 < 0 \) and \( \beta_0 > 1 \), which is identical to the E-stability condition of the AR(1) solution under least squares learning except that \( |\frac{1-\beta_0}{\beta_1}| < 1 \) is needed for the stationarity of the AR(1) solution.

**Example 5.1 (An overlapping generations model with fiat money)**

The overlapping generations model with fiat money in Duffy (1994) corresponds to the model class (54) with \( \alpha = 1 \) and \( \beta_0 = -\beta_1 > 1 \) except that our model is a stochastic version of his and has an i.i.d innovation \( v_t \); see his equation (10).

Suppose agents form expectations using FLDD. Part (b) and (c) of proposition 5.1 say that the MSV solution is not learnable and the AR(1) solution is learnable under the FLDD, respectively. Using an alternative adaptive learning algorithm the result here reinforces the result of Duffy (1994) that the AR(1) solution will be selected under least squares learning. This can be used to explain the sluggish adjustment of price level to monetary disturbances as documented in the data and highlighted in Duffy (1994).

The stability condition of the REE under FLDD when agents’ subjective models are correctly specified is identical to the E-stability condition of the REE under least squares learning in some model classes, see proposition 4.2, 4.3, and 4.4. Note in these model classes, the E-stability condition under least squares learning is identical to the strong E-stability condition under least squares learning. In some model class, the stability condition under FLDD is stronger than the E-stability condition, see part (b) of proposition 5.1, and identical to the strong E-stability condition.

Different stability condition under FLDD from the E-stability condition under least squares learning seems to require that the model is dated with time \( t - 1 \) and has more than one expectational lead. A conjecture is that there is a correspondence between the stability condition of the REE under FLDD and the strong E-stability condition because estimating an over-parameterized model is usually needed to obtain a consistent estimate of \( \rho \). When the E-stability condition differs from the stability condition under FLDD may depend on when the strong E-stability condition is different from the E-stability condition under least squares learning. It seems there exists no general answer to the latter question in the literature and worth further investigation.
6 Conclusion

The paper goes beyond least squares learning and assumes economic agents learn with (quasi-)differencing data as econometricians very often do when perceiving persistent residuals (including stochastic trends as a special case). The paper derives the recursive formulation of the learning algorithms, enabling the study of a wide range of issues in adaptive learning.

I show that the *E-stability principle* is valid for analyzing the convergence of FLDD of both correctly specified and underparameterized models to the REE. Nevertheless, the E-stability condition of a REE under FLDD can be identical or different from the E-stability condition of the REE under least squares learning.

I provide a new perspective on the stability of a REE by establishing the convergence of learning underparameterized models to the REE. The associated convergence condition can be used to inform the desirability of different ways to address model misspecifications in econometric practice.

The business cycle with adaptive learning literature considers only learning about detrended variables and assumes economic agents have exact knowledge about the long-run growth of endogenous variables. The paper relaxes the informational assumption and allows for learning about the (long-run) growth of endogenous variables. The learning process with estimating trends requires a different condition to ensure the convergence and generates more plausible quantitative properties. This suggests the importance of taking into account uncertainty and learning about the (long-run) growth of endogenous variables in dynamic macroeconomic models.\(^{35}\)

An issue associated with adopting more sophisticated methods or allowing for a richer structure of the variance-covariance matrix of the residuals is that agents’ econometric specification is flexible and more general such that it can nest RE equilibria with different functional forms. Studying the convergence and stability of the RE equilibria under FLDD gives a prediction about the RE equilibrium selected by the more sophisticated method.

I provide a new perspective about the stability of the rational bubble solutions by considering a natural expectation formation method (i.e., learning about the growth rate of the bubbly variable), reinforcing the existing results on the fragility of the bubble solutions.

The paper contributes to the literature on expectations-driven fluctuations by demonstrating that beliefs about persistent residuals (including random walk beliefs as a special case) can be nearly self-fulfilling under (feasible) learning with data (quasi-)differencing. This can be used to explain some salient features of fluctuations in asset prices, inflation and macroeconomic activities, such as excess volatilities and regime switching in persistence and

\(^{35}\)See Kuang and Mitra (2014) for an alternative quantitative example.
volatilities.

Finally, the paper provides a caveat on addressing model misspecification in econometric practice by showing that data quasi-differencing or GLS estimation can be exactly self-fulfilling. A simple check of calculating the persistence of residuals cannot detect the problem of wrong estimates relative to the REE. Careful diagnosis or an alternative specification of the model is needed to learn the REE.

References


Appendix (Supplemental Materials)

A Proof of Proposition 2.1

Define

\[ H_a(a) = S^{-1} \lim_{t \to \infty} E[(x_{t-1} - \rho x_{t-2})(y_{t-1} - \rho y_{t-2} - (x_{t-1} - \rho x_{t-2})a)] \]
\[ = S^{-1} E(x_{t-1} - \rho x_{t-2})^2(T_1(a; \rho) - a) \]

Denote \( a^* \) the rest point of the learning process. Setting \( H_a(a) = 0 \) yields \( a^* = a^{RE} = \frac{\delta}{1 - ad} \).

Stability of the REE requires that the partial derivative \( \frac{\partial H_a(a)}{\partial a} \bigg|_{a=a^{RE}} = T_1' \bigg|_{a=a^{RE}} - 1 = \frac{-\alpha(p-d)}{1 - \alpha \rho} - 1 = \frac{\alpha d - 1}{1 - \alpha \rho} < 0 \), which is equivalent to (1) \( ad < 1 \) and \( \alpha \rho < 1 \); or (2) \( ad > 1 \) and \( \alpha \rho > 1 \).

B Proof of Proposition 2.2

Suppose the PLM is \( \Delta y_t = a + \eta_t \). The conditional expectations are \( E_t y_{t+1} = y_t + a \).

The ALM under learning\(^{36}\) is \( y_t = \alpha(y_t + a) + \delta x_t + \nu_t = \frac{\alpha a}{1 - \alpha} + \frac{\delta}{1 - \alpha} x_t + \nu_t \). Differentiating the above equation yields \( \Delta y_t = \frac{\delta}{1 - \alpha} \Delta x_t + \Delta \nu_t = \frac{\delta}{1 - \alpha} \xi_t + \Delta \nu_t \). The learning algorithm is \( a_t = a_{t-1} + \frac{1}{T} (\Delta y_{t-1} - a_{t-1}) \).

Define \( H_a(a) = E(\Delta y_{t-1} - a) = -a \). Setting \( H_a(a) = 0 \) yields the rest point \( a^* = 0 \). Stability requires \( \frac{\partial H_a(a)}{\partial a} \bigg|_{a=a^*} = -1 < 0 \), which is true.

C Proof of Proposition 2.3

Suppose agents’ PLM is \( \log y_t - \log y_{t-2} = a + \eta_t \). Agents’ conditional expectations are \( E_t y_{t+1} = e^{a_t} y_{t-1} \) assuming vanishing variance of \( \eta_t \) perceived by agents. The ALM under learning is

\[ y_t = \alpha e^{a_t} y_{t-1} + x_t + \nu_t \]
\[ = \alpha^2 e^{(a_t + a_{t-1})} y_{t-2} + \alpha e^{a_t} x_{t-1} + \alpha e^{a_t} \nu_{t-1} + x_t + \nu_t \]

\(^{36}\)I rule out \( \alpha = 1 \) in what follows because the T-map is not defined in this case.
The learning algorithm is \( a_t = a_{t-1} + \frac{1}{t} (\log y_{t-1} - \log y_{t-3} - a_{t-1}) \). Substituting the ALM for \( \log y_{t-1} \) yields

\[
a_t = a_{t-1} + \frac{1}{t} (\log \left( \alpha^2 e^{(a_{t-1}+a_t)} y_{t-3} + \alpha e^{a_{t-1}} x_{t-2} + \alpha e^{a_{t-1}} \nu_{t-2} + x_{t-1} + \nu_{t-1} \right) - \log y_{t-3} - a_{t-1})
\]

Fix the subjective estimate \( a \) and define

\[
H_a(a) \equiv \lim_{t \to \infty} E \left( \log \left( \alpha^2 e^{2a} y_{t-3} + \alpha e^a x_{t-2} + \alpha e^a \nu_{t-2} + x_{t-1} + \nu_{t-1} \right) - \log y_{t-3} - a \right)
\]

\[
= \lim_{t \to \infty} E \left( \log \left( \frac{\alpha^2 e^{2a} + \alpha e^a x_{t-2} + \alpha e^a \nu_{t-2} + x_{t-1} + \nu_{t-1}}{y_{t-3}} \right) - a \right)
\]

\[
= 2 \log \alpha + a
\]

Setting \( H_a = 0 \) yields \( a^* = -2 \log \alpha \), so agents’ belief about the growth rate of \( y_t \) converges to the RE value. Since \( \frac{\partial H_a}{\partial a} \big|_{a=a^*} = 1 > 0 \), the rest point is unstable.

**D Proof of Proposition 3.1**

Proof of part (a). The ALM is \( y_t = T_1(a)y_{t-1} + T_2(a)v_t \). Define \( M_0(a) \equiv E[y_t(a)^2] \) and \( M_1(a) \equiv E[y_t(a)y_{t-1}(a)] \). It can be shown that

\[
M_0(a) = \frac{[T_2(a)]^2 \sigma_v^2}{1 - (T_1(a))^2} \quad (62)
\]

\[
M_1(a) = \frac{T_1(a)[T_2(a)]^2 \sigma_v^2}{1 - (T_1(a))^2} \quad (63)
\]

Define

\[
H_S(a, S) = \lim_{t \to \infty} E \left[ (y_{t-1} - \rho y_{t-2})^2 - S \right]
\]

\[
= (1 + \rho^2)M_0(a) - 2\rho M_1(a) - S
\]

\[
= \frac{(\rho^2 - 2\rho T_1(a) + 1) T_2^2(a) \sigma_v^2}{1 - T_1^2(a)} - S
\]
and

\[
H_a(a, S) = S^{-1} \lim_{t \to \infty} E[(y_{t-2} - \rho y_{t-3})(y_{t-1} - \rho y_{t-2} - (y_{t-2} - \rho y_{t-3})a)]
\]
\[
= S^{-1} \lim_{t \to \infty} E[(y_{t-2} - \rho y_{t-3})(T_1(a)y_{t-2} + T_2(a)v_{t-1} - \rho y_{t-2}) - (y_{t-2} - \rho y_{t-3})^2a)]
\]
\[
= S^{-1} E[(T_1(a) - \rho)(y_{t-2}^2 - \rho y_{t-2}y_{t-3} - a(y_{t-2} - \rho y_{t-3})^2)
\]
\[
= S^{-1} M_0(a)[(T_1(a) - \rho) - \rho(T_1(a) - \rho)T_1(a)]
\]
\[
a(1 - 2\rho T_1(a) + \rho^2)
\]

(64)

Given \(\rho\), the learning process will converge to rest points \((a^*, S^*)\) such that \(H_S(a^*, S^*) = 0\) and \(H_a(a^*, S^*) = 0\) or equivalently

\[
(T_1(a^*; \rho) - \rho) - \rho(T_1(a^*; \rho) - \rho)T_1(a^*; \rho) - a^*(1 - 2\rho T_1(a^*; \rho) + \rho^2) = 0
\]

(65)

\[
\frac{(\rho^2 - 2\rho T_1(a^*; \rho) + 1)T_2(a^*; \rho)\sigma_v^2}{1 - T_1(a^*; \rho)^2} = S^*
\]

(66)

Solving equation (65) yields \(a^*\) as a function of \(\rho\). Then \(S^*\) can be determined by equation (66).

The stability of the rest points is determined by the stablity of the matrix

\[
\left( \begin{array}{cc}
\frac{\partial H_a(a^*, S^*)}{\partial a} & \frac{\partial H_a(a^*, S^*)}{\partial S} \\
\frac{\partial H_S(a^*, S^*)}{\partial a} & \frac{\partial H_S(a^*, S^*)}{\partial S}
\end{array} \right)
\]

Note \(\frac{\partial H_a(a^*, S^*)}{\partial S} = 0\) and \(\frac{\partial H_S(a^*, S^*)}{\partial S} = -1\). So the learnability requires \(\frac{\partial H_a(a^*, S^*)}{\partial a} < 0\), which is equivalent to \((1 - 2\rho T_1(a^*) + \rho^2 + 2a^*)\frac{dT_1(a^*)}{da}\big|_{a=a^*} < (1 - 2\rho T_1(a^*) + \rho^2)\) and different from the E-stability condition of the REE under least squares learning \(\frac{dT_1}{da}|_{a=a^{RE}} < 1\) where \(a^{RE}\) denotes the RE coefficient in the REE.

Proof of part (b) is sketched in the text. Proof of Part (c) is as follows. Equation (65) defines an implicit function \(a^*\) of \(\rho\). Taking derivative of both sides of equation (65) with respect to \(\rho\) delivers

\[
\frac{dH_a(a^*, S^*)}{d\rho} = \left( \frac{dT_1(a^*)}{d\rho} - 1 \right) - [(T_1(a^*) - \rho)T_1(a^*) + \rho \left( \frac{dT_1(a^*)}{d\rho} - 1 \right) T_1(a^*)
\]
\[
+ \rho (T_1(a^*) - \rho) \frac{dT_1(a^*)}{d\rho}] - \frac{da^*}{d\rho} (1 - 2\rho T_1 + \rho^2) - a^* \left( 1 - 2T_1(a^*) + 2\rho - 2\rho \frac{dT_1(a^*)}{d\rho} \right)
\]
\[
= 0
\]

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where \( \frac{dT_1(a^*)}{d\rho} \) is the total derivative of \( T_1 \) with respect to \( \rho \). Note \( a^* = 0, \rho = \alpha^R E \), and \( T_1(a^*) = \alpha^R E \). Evaluating the above equality at \( a^* = 0 \) and \( \rho = \alpha^R E \) yields

\[
\frac{d(T_1(a^*) - a^*)}{d\rho} \bigg|_{a^* = 0, \rho = \alpha^R E} = 1
\]

The derivative

\[
\frac{dT_1(a^*)}{d\rho} \bigg|_{a^* = 0, \rho = \alpha^R E} = \left[ \frac{\partial T_1(a^*)}{\partial a} \frac{\partial a^*}{\partial \rho} + \frac{\partial T_1(a^*)}{\partial \rho} \right] \bigg|_{a^* = 0, \rho = \alpha^R E}
\]

Note \( \frac{\partial T_1(a^*)}{\partial a} \bigg|_{a^* = 0, \rho = \alpha^R E} = 0 \), so

\[
\frac{dT_1(a^*)}{d\rho} \bigg|_{a^* = 0, \rho = \alpha^R E} = \frac{\partial T_1(a^*)}{\partial \rho} \bigg|_{a^* = 0, \rho = \alpha^R E} = \frac{\alpha^R E}{1 - \alpha^R E}
\]

The last equation uses the expression of the \( T_1 \)-map.

**E  Proof of Proposition 3.2**

From equation (24), I get \( \text{var}(w_t) = \text{var}(w_{t-1}) = (T_1(a^*) - a^*)^2 M_0(a^*) + T_2(a^*)^2 \sigma_v^2 \) and

\[
cov(w_t, w_{t-1}) = E[((T_1(a^*) - a^*)y_{t-1} + T_2(a^*)v_t) ((T_1(a^*) - a^*)y_{t-2} + T_2(a^*)v_{t-1})]
\]

\[
= (T_1(a^*) - a^*)^2 M_1(a^*) + (T_1(a^*) - a^*)T_2(a^*)^2 \sigma_v^2
\]

The serial correlation of residuals in the ALM is

\[
\rho_w \equiv \frac{\text{cov}(w_t, w_{t-1})}{\sqrt{\text{var}(w_t)\text{var}(w_{t-1})}} = \frac{(T_1(a^*) - a^*)^2 M_1(a^*) + (T_1(a^*) - a^*)T_2(a^*)^2 \sigma_v^2}{(T_1(a^*) - a^*)^2 M_0(a^*) + T_2(a^*)^2 \sigma_v^2}
\]  \tag{67}

Substituting the expressions for \( M_0(a^*) \) and \( M_1(a^*) \) (equation (62) and (63)) into (67) yields that \( \rho_w(\rho^*) = \rho^* \) is equivalent to equation (25).
Proof of Proposition 4.1

The stability of the rest point of the learning process is determined by the stability of the ODE $\frac{\partial H}{\partial a} = H(\theta) = (H_a(\theta) \ H_{c,j}(\theta'))$. Define

$$H_a = \lim_{t \to \infty} E \left[ S^{G-1}(1 - j)(y_{t-1} - jy_{t-2} - a(1 - j)) \right]$$

$$= \lim_{t \to \infty} E \left[ S^{G-1}(1 - j)(T_1(a) + (T_2(a) - j)y_{t-2} - a(1 - j)) \right]$$

$$= S^{G-1}(1 - j)(T_1(a) + (T_2(a) - j)\frac{T_1(a)}{1 - T_2(a)} - a(1 - j))$$  \hspace{1cm} (68)$$

$$H_{SO} = (1 - j)^2 - S^G$$

$$H_{c,j} = \lim_{t \to \infty} ES^{O-1} \left[ \left( \begin{array}{c} 1 \\ y_{t-2} \end{array} \right) (T_1(a) + T_2(a) y_{t-2} - c - jy_{t-2}) \right]$$

$$= \lim_{t \to \infty} ES^{O-1} \left[ \left( \begin{array}{c} 1 \\ y_{t-2} \end{array} \right) \left( \begin{array}{c} 1 \\ T_2(a) \end{array} \right) \left( \begin{array}{c} T_1(a) - c \\ T_2(a) - j \end{array} \right) \right]$$  \hspace{1cm} (69)$$

$$H_{SO} = \left( \begin{array}{c} 1 \\ \frac{T_1(a)}{1 - T_2(a)} \end{array} \right) \left( \begin{array}{c} T_1(a) \\ 1 - T_2(a) + T_2(a) \end{array} \right) - S^O$$

Setting $H_a = 0$ and $H_{c,j} = 0$ yields $T_1(a) = c$ and $T_2(a) = j$ by (69). The latter implies that $j = T_2(a) = a^{RE}$. By (68), I get $T_1(a) = a(1 - j)$, which implies further $a^* = 0$ with the expression of the $T_1$-map.

Denote “$|_{RE}$” evaluating the derivative at the REE. Learnability of the REE requires

1) $\frac{\partial H_a}{\partial a}|_{RE} = S^{G-1}(1 - j) \left[ \frac{\partial T_1}{\partial a} - (1 - j) \right]|_{RE} < 0$. This amounts to $\frac{\alpha}{1 - \alpha a^{RE}} < 1$ assuming $a^{RE} < 1$. Note this condition is identical to $\frac{d}{da} \left( \frac{T_1(a)}{1 - T_2(a)} \right)|_{RE} = \frac{\partial T_1}{\partial a}|_{RE} < 1$. 2) $\frac{\partial T_2}{\partial c}|_{RE} < 0$, which is satisfied automatically. 3) $\frac{\partial T_2}{\partial j}|_{RE} < 1$, which amounts to the E-stability condition $\frac{\alpha a^{RE}}{1 - \alpha j} < 1$. In addition, the stationarity of the ALM requires that $|T_2| = \left| \frac{\delta}{1 - \alpha j} \right| < 1$. To summarize, the learnability condition is $\frac{\alpha}{1 - \alpha a^{RE}} < 1$, the E-stability condition $\frac{\alpha a^{RE}}{1 - \alpha a^{RE}} < 1$ and $\left| \frac{\delta}{1 - \alpha a^{RE}} \right| < 1$. 

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G  Proof of Proposition 4.2

Define \( y_t^* = y_t - \rho y_{t-1} \) and \( x_t^* = x_t - \rho x_{t-1} \). Equation (32)-(33) can be transformed to

\[
y_t^* = ay_t^{*} + bx_t^{*} + \omega_t
\]

(70)

Define \( \Sigma_{y^*} = Var(y_{t-1}^*) \), \( \Sigma_{y^*x^*} = Cov(y_{t-1}^*, x_{t-1}^*) \), and \( \Sigma_{x^*} = Var(x_{t-1}^*) \). The stability of the REE is determined by stability of the following ODEs

\[
H_{a,b} = \lim_{t \to \infty} S^{G-1}E \left[ \begin{pmatrix} y_{t-2}^* \\ x_{t-2}^* \end{pmatrix} \begin{pmatrix} y_{t-1}^* - ay_{t-2}^* - bx_{t-2}^* \end{pmatrix} \right]
\]

(71)

\[
= \lim_{t \to \infty} S^{G-1}E[\begin{pmatrix} y_{t-2}^* \\ x_{t-2}^* \end{pmatrix} \begin{pmatrix} (T_1(a) - a)y_{t-2}^* + (T_2(a) - b)x_{t-2}^* - \rho T_3(a)A_{t-2} \end{pmatrix}]
\]

(72)

\[
= S^{G-1} \begin{pmatrix} \Sigma_{y^*} & \Sigma_{y^*x^*} \\ \Sigma_{x^*y^*} & \Sigma_{x^*} \end{pmatrix} \begin{pmatrix} T_1(a) - a \\ T_2(a) - b \end{pmatrix} - S^{G-1}\rho E \begin{pmatrix} y_{t-2}^*T_3(a)A_{t-2} \\ x_{t-2}^*T_3(a)A_{t-2} \end{pmatrix}
\]

(73)

Denote \( \tilde{z}_{t-2} = (y_{t-2}, y_{t-3}, x_{t-2}, x_{t-3}) \), \( \tilde{T} = (T_1(a) - c - e, T_2(a) - f - g) \), and \( H_{c,e,f,g} = E[S^{G-1} \tilde{z}_{t-2} \tilde{T}] \).

Setting \( H_{a,b} = 0 \) and \( H_{c,e,f,g} = 0 \) yields the rest point \( e = 0 \), \( g = 0 \), \( T_1(a) = c \), and \( T_2(a) = f \). This implies \( \rho = -\frac{a}{b} = 0 \). So the rest point is the REE. Second, learnability requires that \( \frac{\partial T_1}{\partial a} \big|_{RE} < 1 \), \( \frac{\partial T_2}{\partial b} \big|_{RE} < 1 \), and \( \frac{\partial T_3}{\partial f} \big|_{RE} < 1 \) where \( \frac{\partial}{\partial |RE} \) means evaluating the derivatives at the REE. Evaluating at the REE, \( \frac{\partial T_1}{\partial c} \big|_{RE} = \frac{\partial T_3}{\partial c} \big|_{RE} = \frac{\partial T_3}{\partial f} \big|_{RE} = 0 \). Therefore, the learnability condition coincides with the E-stability condition under least squares learning.

H  Proof of Proposition 4.3

The stability of the learning process (43) – (47) is determined by the stability of \( \frac{\partial}{\partial \tau} = (H_{a,c} H_{c,m}) \) where

\[
H_{c,m} = E[S^{G-1} \begin{pmatrix} y_{t-3} \\ y_{t-4} \end{pmatrix} \begin{pmatrix} (T_1(a^G) - c)y_{t-3} - my_{t-4} \end{pmatrix}]
\]

\[
= S^{G-1} \frac{T_2(a^G)}{1 - T_2^2(a^G)} \begin{pmatrix} T_1(a^G) - c - mT_1 \\ T_1(a^G)(T_1(a^G) - c) - m \end{pmatrix}
\]

(74)

And \( H_{a,c} \) is still equation (64). Setting \( H_{a,c} = 0 \) and \( H_{c,m} = 0 \) yields \( \rho = 0 \), \( a^G = a^{RE} = c \), and \( m = 0 \). Note \( \frac{\partial T_1}{\partial a} \big|_{\rho=0,a^G=a^{RE}=c,d=0} = \frac{aa}{1-aa} \), \( \frac{\partial T_1}{\partial \rho} \big|_{\rho=0,a^G=a^{RE}=c,m=0} = 0 \). The Jacobian matrix
is
\[
\begin{pmatrix}
\frac{\partial T_1}{\partial a^G}|_{\rho=0,a^G=a^{RE},c,m=0} - 1 & 0 & \Delta_7 \\
\Delta_8 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]
where $\Delta$'s are non-zero elements. The eigenvalues are on the diagonal of this matrix. The learnability condition in the case is identical to the E-stability condition under least squares learning.

I \hspace{1em} \textbf{Proof of Proposition 4.4}

The stability of the learning process is determined by the stability of the associated ODEs, 
\[
\frac{\partial \theta}{\partial t} = H(\theta) = (H_{a^G}(\theta) \ h_{a^O}(\theta) \ H_{a^O}(\theta) \ H_{S_p}(\theta)).
\]
Define $M_{x,0} = E(x_t^2)$ and $M_{x,1} = E(x_t x_{t-1}).$ The right hand side (RHS) of the ODEs are
\[
\begin{align*}
H_{a^G}(\theta) &= S^{G-1}(M_{x,0}(1 + \rho^2) - 2\rho M_{x,1})(T_1(a^G; \rho) - a^G) & (74) \\
H_{SC}(\theta) &= (M_{x,0}(1 + \rho^2) - 2\rho M_{x,1}) - S^G & (75) \\
H_{a^O}(\theta) &= S^{O-1}(T_1(a^G; \rho) - a^O) & (76) \\
H_{SO}(\theta) &= M_{x,0} - S^O & (77) \\
H_{p}(\theta) &= S_p^{-1}[(M_{x,1} - \rho M_{x,0})(T_1(a^G; \rho) - a^O)^2 - \rho T_2(a^G; \rho)] & (78) \\
H_{S_p}(\theta) &= (T_1(a^G; \rho) - a^O)^2 M_{x,0} + T_2(a^G; \rho) - S_p & (79)
\end{align*}
\]
Setting all the H functions to zero, I get $T_1(a^{G*}; \rho^*) = a^{O*}$ from (76). Together with (78), I obtain $\rho^* = 0.$ Then (74) and (76) imply that $a^{O*} = a^{G*} = a^{RE}.$ So agents' beliefs about $a^G$ and $\rho$ will converge to the RE values.

Evaluated at the RE equilibrium, the Jacobian of the RHS of the ODEs is
\[
\begin{pmatrix}
T_1'(a^{RE}; 0) - 1 & 0 & 0 & 0 & \Delta_1 & 0 \\
0 & -1 & 0 & 0 & \Delta_2 & 0 \\
\Delta_3 & 0 & -1 & 0 & \Delta_4 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -T_2^2(a^{RE}; 0) & 0 & 0 \\
\Delta_5 & 0 & 0 & 0 & \Delta_6 & -1
\end{pmatrix}
\]

The $\Delta$'s above are non-zero elements. Their expressions are omitted because they don't matter for the calculation of eigenvalues of the Jacobian matrix. Stability of the rest point requires all eigenvalues to be negative. Note the eigenvalues are on the diagonal. I obtain that
the stability condition under FLDD is \( T'_1(a^{RE}; 0) < 1 \), which is identical to the E-stability condition under least squares learning.

**J Proof of Proposition 5.1**

Proof of part (b). Consider the MSV REE with \( a^G = a^{RE} \) and \( \rho = 0 \). The ALM under learning is \( y_t = T_1(a^G) + T_2(a^G)y_{t-1} + v_t \).

\[
H_{a^G} = \lim_{t \to \infty} ES^{G-1}(1 - \rho)(y_{t-1} - \rho y_{t-2} - a^G(1 - \rho))
\]

\[
= S^{G-1}(1 - \rho) (T_1(a^G) + (T_2(a^G) - \rho) E(y_{t-2}) - a^G(1 - \rho))
\]

Denote \( \mid_{RE} \) evaluating at the MSV RE solution. Note \( \frac{\partial H_{a^G}}{\partial a^G} \mid_{RE} < 0 \) is equivalent to \( \beta_0 + \beta_1 < 1 \), which is identical to \( \frac{\partial T_1}{\partial \rho} \mid_{RE} < 1 \). Define

\[
H_{a^G} = \lim_{t \to \infty} ES^{O-1}(1 - \rho)(y_{t-2}) (T_1 - n + (T_2 - \rho)y_{t-2})
\]

\[
= S^{O-1} \begin{pmatrix}
1 \\
y_{t-2}
\end{pmatrix} (T_1 - n)
\]

Learning about the persistence requires additionally \( \frac{\partial T_2}{\partial \rho} \mid_{RE} = \beta_0 - 1 < 0 \), which is identical to \( \beta_0 < 1 \).

Proof of part (c). Define

\[
H_{a^G} = \lim_{t \to \infty} ES^{G-1}(1 - \rho)(y_{t-1} - \rho y_{t-2} - a^G(1 - \rho))
\]

\[
= S^{G-1}(1 - \rho) (T_1(a^G) + (T_2(a^G) - \rho) E(y_{t-2}) - a^G(1 - \rho))
\]

and

\[
H_{a^G} = ES^{O-1}(1 - \rho)(y_{t-2}) (T_1(a^G) - n + (T_2(a^G) - \rho)y_{t-2})
\]

The AR(1) solution is a rest point of the learning process with \( a^G = a^{AR} = -\frac{\beta_0 \rho}{1 - \rho} \) and \( \rho = \beta_1^{-1}(1 - \beta_0) = T_2(a^{AR}) \).

Denote \( \mid_{RE} \) evaluating at the AR(1) RE solution with \( a^G = a^{AR} \) and \( \rho = \beta_1^{-1}(1 - \beta_0) \).
Note \( \frac{\partial T_1}{\partial \alpha} |_{AR} = (\beta_0 (1 - \rho) + \beta_1 (1 - \rho^2)) |_{AR} \) and \( \frac{\partial T_2}{\partial \alpha} |_{AR} = 0 \). Stability under FLDD requires

\[
\frac{\partial H_{\alpha G}}{\partial \alpha G} |_{AR} = \frac{1}{1 - \rho} \left( \frac{\partial T_1 (a^G)}{\partial \alpha G} - (1 - \rho) + \frac{\partial T_2 (a^G)}{\partial \alpha G} \frac{T_1 (a^G)}{1 - T_2 (a^G)} \right) |_{AR} < 0
\]

\( \iff \beta_1 < 0 \)

\( \frac{\partial T_1}{\partial \eta} |_{AR} - 1 < 0 \) and \( \frac{\partial T_2}{\partial \rho} |_{AR} - 1 < 0 \). Note \( \frac{\partial T_1}{\partial \eta} |_{AR} = 0 \) and \( \frac{\partial T_2}{\partial \rho} |_{AR} = 2 - \beta_0 \). To summarize, stability requires \( \beta_0 > 1 \) and \( \beta_1 < 0 \). Stationarity of the AR(1) solution requires \(|T_2| = \left| \frac{1 - \beta_0}{\beta_1} \right| < 1\).