Capital Requirements, Risk Measures and Optimal Capital Injections

Walter Farkas
University of Zurich and ETH Zurich

Pablo Koch-Medina
Swiss Reinsurance Company

Cosimo-Andrea Munari
ETH Zurich

Birmingham, 25. January 2012

Financial institutions and capital adequacy

- *Liability holders* of a financial institution are credit sensitive: they, and *regulators* on their behalf, are concerned that the institution may fail to honor its future obligations.
- This will be the case if the institution's *financial position*, i.e. the value of its assets less the value of its liabilities, becomes negative in some future state of the world.
- To address this concern financial institutions hold *risk capital*, which is meant to absorb unexpected losses thereby reducing the likelihood that they may become insolvent.

- A key question is how much capital a financial institution should be required to hold to be deemed *adequately capitalized* by the regulator.

Capital adequacy and acceptance sets

- The previous question can be usefully framed from a mathematical point of view using the concept of an *acceptance set*.
- Let Ω be a given *sample space* representing the set of all possible *future states* of the world.
- Let $\mathscr{X}$ be the set of all possible net financial positions $X : \Omega \to \mathbb{R}$ of a given financial institution, i.e. $X = A - L$, where the random variables $A : \Omega \to \mathbb{R}$ and $L : \Omega \to \mathbb{R}$ represent the value of the *assets* and *liabilities* respectively.
- An acceptance set $\mathscr{A}$ is a subset of $\mathscr{X}$ whose elements represent all financial positions that are deemed to provide a reasonable security to the liability holders.
- Hence, testing whether a financial institution is *adequately capitalized* or not reduces to establishing whether its financial position belongs to $\mathscr{A}$ or not.
- *Coherent acceptance sets* were introduced in 1999 in the seminal paper by Artzner, Delbaen, Eber and Heath for finite sample spaces and in Delbaen (2000) for general probability spaces.
- This setting was extended to *convex acceptance sets* in Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002).

Risk measures and capital requirements

- A fundamental question to ask is: can certain actions by the management of a financial institution *turn an unacceptable financial position into an acceptable one* and *at which cost*?

- The standard theory of risk measures was designed to answer this question for a very particular case: can a badly capitalized financial institution be re-capitalized to acceptable levels by raising additional capital and investing it in a *pre-specified risk-free traded asset*?

- More generally, an investment in a *class of fixed tradable assets* leads to a *risk measure* with respect to these assets: it represents the minimum amount of capital, the so-called *required capital*, that needs to be injected in order to make a position acceptable.

Effectiveness, robustness, efficiency

In our papers we discuss the dependency of required capital on the choice of the acceptance set and the class of tradable assets, and we study the main properties of the corresponding risk measures.

Our three driving questions are therefore the following:

1. is required capital a well-defined number for any financial position?

   This is related to the *effectiveness* of the choice of the eligible assets, i.e. to the ability to make any unacceptable position acceptable.

2. is required capital a continuous function of financial positions?

   This is related to the *robustness* of the required capital figure, i.e. to the reliability of required capital in a world of estimates.

3. can the eligible assets be chosen in such a way that for every financial position the corresponding required capital is lower than if any other class had been chosen?

   This is related to the *efficiency* of the choice of the eligible assets, i.e. to the ability to reach acceptance with the least possible amount of capital.

References

The main references can be downloaded at these links:

- http://ssrn.com/abstract=1966645
- http://ssrn.com/abstract=1989077

Financial positions

We will set our discussion in the following framework. We consider:

- a single-period economy with dates $t = 0$ and $t = 1$;
- a sample space Ω (not necessarily finite) endowed with a σ-algebra $\mathcal{F}$, representing uncertainty at time $t = 1$;
- the Banach space $\mathscr{X}$ of bounded random variables of the form $X : \Omega \to \mathbb{R}$ equipped with the supremum norm

$$\|X\| := \sup_{\omega \in \Omega} |X(\omega)| \;;$$

- the partial order $\leq$ on $\mathscr{X}$ such that $X \leq Y$ whenever $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$;
- the corresponding positive cone $\mathscr{X}^+ := \{X \in \mathscr{X} \,;\, X \geq 0\}$, whose interior is non-empty and consists of those positions X that are *bounded away from zero*, i.e. such that for some $\varepsilon > 0$ we have $X \geq \varepsilon 1_\Omega$, with $1_\Omega(\omega) = 1$ for any $\omega \in \Omega$.

Assumption

- The net *financial position* of a financial institution at time $t = 1$ will be represented as a random variable in the space $\mathscr{X}$.
- We will require a *probabilistic model* on $(\Omega, \mathcal{F})$ only for some specific examples. In this sense, most of the next results will be “model free”.

Acceptable financial positions

Definition

A set $\mathscr{A} \subset \mathscr{X}$ is called an *acceptance set* whenever the following two conditions are satisfied:

(A1) $\mathscr{A}$ is a non-empty, proper subset of $\mathscr{X}$ (non-triviality);

(A2) if $X \in \mathscr{A}$ and $Y \geq X$ then $Y \in \mathscr{A}$ (monotonicity).

Remark

An acceptance set encapsulates the minimum requirements we would place on a concept of non-trivial *capital adequacy test*:

- some – but not all – positions should be acceptable;
- any financial position dominating an already accepted position should also be acceptable.

Definition

A financial institution with net financial position $X \in \mathscr{X}$ will be said to be *adequately capitalized* with respect to $\mathscr{A}$ if $X \in \mathscr{A}$.

Coherent acceptance sets

Introduced by Artzner, Delbaen, Eber and Heath (1999), coherent acceptance sets $\mathscr{A} \subset \mathscr{X}$ are *convex cones*. As such for any $X, Y \in \mathscr{A}$ and any $\lambda \in [0,1]$

$$\lambda X + (1-\lambda)Y \in \mathscr{A} \quad \text{(convexity)}\ ,$$

and for any $X \in \mathscr{A}$ and any $\lambda \geq 0$

$$\lambda X \in \mathscr{A} \quad \text{(cone)}\ .$$

The most known and used example of coherent acceptance set is the acceptance set based on *Expected Shortfall* at level $\alpha \in (0,1)$

$$\mathscr{A}^\alpha := \left\{ X \in \mathscr{X}\ ;\ \mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta \leq 0 \right\}$$

where $\mathrm{VaR}_\alpha(X)$ is the *Value-at-Risk* at level α of the position $X \in \mathscr{X}$ defined by

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}\ ;\ \mathbb{P}(X+m<0) \leq \alpha\}\ .$$

Here $\mathbb{P}$ is a given probability measure on the measurable space $(\Omega, \mathcal{F})$.

Convex acceptance sets

Convex acceptance sets were introduced by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). Being convex, any coherent acceptance set is also a convex acceptance set, while the converse is not true.

Genuine convex acceptance sets are for example the ones associated with a non-empty collection $\mathscr{Q}$ of finitely additive set functions

$$\mathbb{Q} : \mathcal{F} \to [0, 1]$$

with $\mathbb{Q}(\Omega) = 1$ and with a *floor*

$$\gamma : \mathscr{Q} \to \mathbb{R}$$

such that $\sup_{\mathbb{Q} \in \mathscr{Q}} \gamma(\mathbb{Q}) < \infty$, and having the form

$$\mathscr{A}(\mathscr{Q}, \gamma) := \bigcap_{\mathbb{Q} \in \mathscr{Q}} \{X \in \mathscr{X} \, ; \, \mathbb{E}_{\mathbb{Q}}[X] \geq \gamma(\mathbb{Q})\} \ .$$

Note that the previous condition $\sup_{\mathbb{Q} \in \mathscr{Q}} \gamma(\mathbb{Q}) < \infty$ guarantees that the set $\mathscr{A}(\mathscr{Q}, \gamma)$ is not empty.

Value-at-Risk based acceptance sets

The most widely used acceptance set is the one based on Value-at-Risk at level $\alpha \in (0,1)$ and it is defined by

$$\begin{aligned} \mathscr{A}_\alpha &:= \{X \in \mathscr{X}\,;\, \mathrm{VaR}_\alpha(X) \leq 0\} \\ &= \{X \in \mathscr{X}\,;\, \mathbb{P}(X < 0) \leq \alpha\}\ , \end{aligned}$$

where $\mathbb{P}$ is a given probability measure on the measurable space $(\Omega, \mathcal{F})$ and as above

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}\,;\, \mathbb{P}(X + m < 0) \leq \alpha\}\ .$$

It is well known that $\mathscr{A}_\alpha$ is not convex, henceforth not coherent.

Remark

- In the sequel we will assume neither coherence nor convexity, making the results as general as possible.
- In particular all the following results will apply to Value-at-Risk acceptance.

The class of eligible assets

We allow to modify the acceptability of financial positions by investing in a class of assets $\mathscr{S}$ made of N fixed tradable *eligible assets* $S^1, \cdots, S^N$ with

- initial price $S_0^j > 0$,
- final payoff $S_1^j \in \mathscr{X}$ such that $S_1^j(\omega) > 0$ for all $\omega \in \Omega$.

The corresponding marketed space, i.e. the set of all possible portfolios made of such assets, is given by

$$\mathscr{M}(\mathscr{S}) := \mathrm{span}(\{S_1^1, \cdots, S_1^N\}) = \left\{ \sum_{j=1}^{N} \lambda_j S_1^j \; ; \; \lambda_1, \cdots, \lambda_N \in \mathbb{R} \right\} .$$

Assumption

- There exists a linear pricing functional $\pi : \mathscr{M}(\mathscr{S}) \to \mathbb{R}$ and we set

$$\mathscr{M}_m(\mathscr{S}) := \{ Z \in \mathscr{M}(\mathscr{S}) \, ; \, \pi(Z) = m \} \; ;$$

- Absence of arbitrage: $\mathscr{M}_0(\mathscr{S}) \cap (\mathscr{X}^+ \setminus \{0\}) = \emptyset$.

From unacceptable to acceptable

Definition

We define for an arbitrary set $\mathscr{A} \subset \mathscr{X}$ the function $\rho_{\mathscr{A},\mathscr{S}} : \mathscr{X} \to [-\infty, \infty]$ by setting for any $X \in \mathscr{X}$

$$\rho_{\mathscr{A},\mathscr{S}}(X) := \inf \{\pi(Z) \in \mathbb{R} ; \, Z \in \mathscr{M}(\mathscr{S}) \, : \, X + Z \in \mathscr{A}\} .$$

In case $\mathscr{S} = \{S\}$ we will simply write $\rho_{\mathscr{A},S}$ and we have

$$\rho_{\mathscr{A},S}(X) = \inf \left\{ m \in \mathbb{R} \, ; \, X + \frac{m}{S_0} S_1 \in \mathscr{A} \right\} .$$

Remark

- The quantity $\rho_{\mathscr{A},\mathscr{S}}(X)$ represents, if finite, the least amount of capital to be invested in the class $\mathscr{S}$ that is able to guarantee the acceptability of X.
- If positive, the quantity $\rho_{\mathscr{A},\mathscr{S}}(X)$ can thus be regarded as a *capital requirement*: it defines the *capital required* to turn the position X into an acceptable position, or the cost of making X acceptable.
- If negative, $\rho_{\mathscr{A},\mathscr{S}}(X)$ represents the amount of capital that can be extracted from a well-capitalized financial institution without compromising the acceptability of its financial position X.

Capital requirements and risk measures

The capital requirement $\rho_{\mathscr{A},\mathscr{S}}$ satisfies the two defining properties of a *risk measure*.

Proposition

Fix $\mathscr{A} \subset \mathscr{X}$. *Then*

(i) $\rho_{\mathscr{A},\mathscr{S}}$ *is translation invariant with respect to* $\mathscr{S}$, *i.e. for all* $X \in \mathscr{X}$ *and* $Z \in \mathscr{M}(\mathscr{S})$

$$\rho_{\mathscr{A},\mathscr{S}}(X+Z) = \rho_{\mathscr{A},\mathscr{S}}(X) - \pi(Z) \ .$$

In particular if $\mathscr{S} = \{S\}$ *then* $\rho_{\mathscr{A},S}$ *is translation invariant with respect to* S, *i.e. for all* $X \in \mathscr{X}$ *and* $m \in \mathbb{R}$

$$\rho_{\mathscr{A},S}(X + mS_1) = \rho_{\mathscr{A},S}(X) - mS_0 \ .$$

(ii) *If* $\mathscr{A}$ *satisfies the monotonicity axiom* $(\mathrm{A2})$ *then* $\rho_{\mathscr{A},\mathscr{S}}$ *is monotone, i.e. for any* $X, Y \in \mathscr{X}$ *such that* $X \leq Y$

$$\rho_{\mathscr{A},\mathscr{S}}(X) \geq \rho_{\mathscr{A},\mathscr{S}}(Y) \ .$$

Comparison with existing literature (1)

Our context is more general than the standard setting for risk measures as it is presented in

- Artzner, Delbaen, Eber and Heath (1999),
- Föllmer and Schied (2002),
- Frittelli and Rosazza Gianin (2002).

- We work with *general acceptance sets*. We require neither coherence nor convexity.
- We allow for investments in a *class of assets* rather than one single asset.
- Even in case of a single eligible asset S, we don't require that it is risk-free, i.e. $S_1 = 1_\Omega$.
- As a result we deal with a more *general form of translation invariance* than simple cash-additivity.
- The function $\rho_{\mathscr{A},\mathscr{S}}$ may consequently take both the value ∞ and $-\infty$.
- Even if $\rho_{\mathscr{A},\mathscr{S}}$ is finitely valued, its continuity is not ensured any more, as in Föllmer and Schied (2011).

Comparison with existing literature (2)

Frittelli and Scandolo (2006) introduce capital requirements with respect to a class of assets (see Definition 3.1 and 3.4) and extend the treatment to a multi-period economy. Yet for their main results they always assume the standard one-asset cash-additive framework (from Proposition 4.5 on).

Recent papers study risk measures defined on more general spaces than the space $\mathscr{X}$ of bounded $\mathcal{F}$-measurable random variables of the form $X : \Omega \to \mathbb{R}$, but they still deal with standard cash-additive risk measures which are assumed to be finitely valued, or to take at most the value ∞.

- L^p spaces: Kaina and Rüschendorf (2009)
- Morse spaces: Cheridito and Li (2009)
- Orlicz spaces: Biagini and Frittelli (2008), Arai (2010)
- Fréchet lattices: Biagini and Frittelli (2009)

Comparison with existing literature (3)

Some very recent papers deal indeed with risk measures that are invariant with respect to a single pre-specified asset (whose payoff is often called *numeraire*). Even if they don't require the numeraire to be the risk-free asset, they explicitly or implicitly assume more than just asking that it is *everywhere positive*.

- Filipović and Kupper (2007) deal with ordered normed spaces and are interested in inf-convolutions of convex risk measures. In Lemma 3.5 they assume a condition that in our setting is equivalent to boundedness away from zero of S_1.
- Konstantinides and Kountzakis (2011) work on ordered reflexive spaces with non-empty cone interior, and assume that the numeraire belongs to the interior of the cone generating the partial order: in our setting, this is equivalent to the boundedness away from zero of S_1.
- Kountzakis (2011) extend the previous results on non-reflexive ordered normed spaces with non-empty cone interior, but again asks the numeraire to be bounded away from zero or more generally to be an everywhere positive *quasi-interior* point.

Motivation

This work has been mainly motivated by the following issues.

- The intent of re-emphasizing (as done in Artzner, Delbaen, Eber and Heath (1999)) the *primacy of acceptance sets* with respect to risk measures in a capital adequacy environment: the *raison d'être* of a risk measure is to measure how far a position is from acceptability using a particular eligible asset as a sort of yardstick, which is meaningful only if we already know what acceptability is supposed to mean.
- The intent of dealing with general (non-convex) acceptance sets, exploiting *monotonicity* as the only essential property of acceptability.
- The intent of allowing for *general forms of investments* in order to establish capital requirements.
- The crucial question about the existence of *optimal investments* leading to lower capital requirements than any other investment.

Capital requirements with respect to a single eligible asset

We recall the definition of a capital requirement with respect to a single eligible asset and we list its main properties.

Proposition

Fix an acceptance set $\mathscr{A} \subset \mathscr{X}$ and an eligible asset S with price $S_0 > 0$ and payoff $S_1 \in \mathscr{X}$ such that $S_1(\omega) > 0$ for any $\omega \in \Omega$. We set

$$\rho_{\mathscr{A},S}(X) = \inf\left\{m \in \mathbb{R}\ ;\ X + \frac{m}{S_0}S_1 \in \mathscr{A}\right\} .$$

The basic properties of $\rho_{\mathscr{A},S} : \mathscr{X} \to [-\infty, \infty]$ are the following:

(i) *$\rho_{\mathscr{A},S}$ cannot be identically equal to ∞ or $-\infty$;*

(ii) *$\rho_{\mathscr{A},S}$ is translation invariant with respect to S, i.e. for all $X \in \mathscr{X}$ and $m \in \mathbb{R}$ we have $\rho_{\mathscr{A},S}(X + mS_1) = \rho_{\mathscr{A},S}(X) - mS_0$;*

(iii) *$\rho_{\mathscr{A},S}$ is monotone, i.e. for any $X, Y \in \mathscr{X}$ such that $X \leq Y$ we have $\rho_{\mathscr{A},S}(X) \geq \rho_{\mathscr{A},S}(Y)$.*

Risk measures and capital requirements

We say that a function $\rho : \mathscr{X} \to \mathbb{R}$ is a *risk measure* with respect to a given eligible asset S if it is translation invariant with respect to S, i.e. if for any $X \in \mathscr{X}$ and $m \in \mathbb{R}$

$$\rho(X + mS_1) = \rho(X) - mS_0 \ ,$$

and if it is monotone, i.e. if for any $X, Y \in \mathscr{X}$ such that $X \leq Y$

$$\rho(X) \geq \rho(Y) \ .$$

Proposition

Let ρ be a risk measure with respect to a given eligible asset S. Then

(i) $\mathscr{A}_\rho := \{X \in \mathscr{X} \, ; \, \rho(X) \leq 0\}$ *is an acceptance set;*

(ii) $\rho = \rho_{\mathscr{A}_\rho, S}$.

Remark

Any risk measure of this kind can be therefore expressed as a capital requirement $\rho_{\mathscr{A}, S}$ for a suitable choice of the acceptance set $\mathscr{A}$.

Convex and coherent acceptance sets

If we require that the acceptance set $\mathscr{A}$ is convex or coherent we derive peculiar properties for $\rho_{\mathscr{A},S}$.

Proposition

(i) *If $\mathscr{A}$ is convex and $\rho_{\mathscr{A},S} : \mathscr{X} \to (-\infty, \infty]$ then $\rho_{\mathscr{A},S}$ is convex, i.e. for $X, Y \in \mathscr{X}$ and $\lambda \in [0,1]$ we have*

$$\rho_{\mathscr{A},S}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{\mathscr{A},S}(X) + (1-\lambda)\rho_{\mathscr{A},S}(Y) \ .$$

(ii) *If $\mathscr{A}$ is a cone (in particular if $\mathscr{A}$ is coherent) and $\rho_{\mathscr{A},S}(0) = 0$ then $\rho_{\mathscr{A},S}$ is positively homogeneous, i.e. for all $X \in \mathscr{X}$ and $\lambda \geq 0$*

$$\rho_{\mathscr{A},S}(\lambda X) = \lambda \rho_{\mathscr{A},S}(X) \ .$$

(iii) *If $\mathscr{A}$ is coherent, $\rho_{\mathscr{A},S} : \mathscr{X} \to (-\infty, \infty]$ and $\rho_{\mathscr{A},S}(0) = 0$, then $\rho_{\mathscr{A},S}$ is subadditive, i.e. for $X, Y \in \mathscr{X}$ we have*

$$\rho_{\mathscr{A},S}(X + Y) \leq \rho_{\mathscr{A},S}(X) + \rho_{\mathscr{A},S}(Y) \ .$$

Eligible asset and numeraire asset

The eligible asset is by definition the only asset which is eligible to modify the acceptability properties of a financial position.

It is yet standard procedure to choose the eligible asset S as the *numeraire*, i.e. as the unit of account.

- In this case one does not consider financial positions $X \in \mathscr{X}$ but rather their *discounted* versions $\frac{X}{S_1}$.

In this setting the translation invariance formula takes the simpler form

$$\rho_{\mathscr{A},S}(X+m) = \rho_{\mathscr{A},S}(X) - m \ .$$

Eligible asset and numeraire asset

Even though we might find it occasionally convenient to *use* the eligible asset as the numeraire, we will refrain from adopting this convention for three main reasons:

1. because converting the question of capital adequacy into a problem about discounted values may lead to lose sight on the role of the eligible asset;
2. because it hides the critical dependence on the chosen eligible asset and investing in the eligible asset is just one of the means to make an unacceptable position acceptable. It is useful to remain aware at all times that the eligible asset is indeed a choice;
3. because a theory based on discounted assets on the space of bounded $\mathcal{F}$-measurable functions implicitly assumes that the payoff S_1 of the eligible asset is bounded away from zero, and this assumption critically limits the choice of the eligible asset itself.

When is required capital well defined for all financial positions?

Requiring that $\rho_{\mathscr{A},S}$ is real valued is economically reasonable. Indeed fix a position $X \in \mathscr{X}$.

- If $\rho_{\mathscr{A},S}(X) = \infty$ then X cannot be made acceptable by investing any amount of capital in the eligible asset, suggesting that we should choose a different eligible asset.
- If $\rho_{\mathscr{A},S}(X) = -\infty$, we can extract arbitrary amounts of capital retaining the acceptability of X, suggesting that the particular acceptance set might be too large and therefore might also accept positions leading to huge losses.

Definition

We will say that a given eligible asset S is *well behaved* with respect to a fixed acceptance set $\mathscr{A}$ if the required capital $\rho_{\mathscr{A},S}(X)$ is finite for any position $X \in \mathscr{X}$.

Effectiveness

Proposition

The function $\rho_{\mathscr{A},S}$ assigns a finite value to a position $X \in \mathscr{X}$ if and only if there exists $m_0 \in \mathbb{R}$ such that

$$X + \frac{m}{S_0} S_1 \notin \mathscr{A} \quad \text{if} \quad m < m_0 \quad \text{and} \quad X + \frac{m}{S_0} S_1 \in \mathscr{A} \quad \text{if} \quad m > m_0 \; .$$

If such an m_0 exists, then $m_0 = \rho_{\mathscr{A},S}(X)$.

In general we can find examples of $\mathscr{A}$ and S that allow for non-finite required capital.

Example

Let Ω be infinite and assume that the payoff S_1 is not bounded away from zero.

1. If we set $\mathscr{A} := \mathscr{X}^+$ then $\rho_{\mathscr{A},S}(-1_\Omega) = \infty$.
2. If we set $\mathscr{A} := \{X \in \mathscr{X} \, ; \, \exists \, \omega_0 \in \Omega \, : \, X(\omega_0) > 0\}$ then $\rho_{\mathscr{A},S}(1_\Omega) = -\infty$.

A sufficient condition for effectiveness

We present a sufficient condition on the payoff S_1 of the eligible asset so that S turns out to be well behaved with respect to *any* acceptance set.

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be an arbitrary acceptance set and assume that the payoff S_1 of the eligible asset is bounded away from zero. Then $\rho_{\mathscr{A},S}(X)$ is finite for any position $X \in \mathscr{X}$.

Remark

- Note that the payoff $S_1 := 1_\Omega$ of the risk-free asset is bounded away from zero. Hence, in this case the function $\rho_{\mathscr{A},S}$ is real valued for any acceptance set $\mathscr{A}$.
- Note that in case Ω is finite, S_1 is bounded away from zero since $S_1(\omega) > 0$ for all $\omega \in \Omega$. It follows that for finite Ω the function $\rho_{\mathscr{A},S}$ is real valued for any choice of $\mathscr{A}$ and S. The same happens if, more generally, the σ-algebra $\mathcal{F}$ is finite.

Effectiveness: VaR and ES case

In general the previous condition is not necessary for the finiteness of $\rho_{\mathscr{A},S}$.

Example

Fix $\alpha \in (0,1)$ and a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

1. *Any* eligible asset S is well behaved with respect to the Value-at-Risk based acceptance set
$$\mathscr{A}_\alpha = \{X \in \mathscr{X} \,;\, \mathbb{P}(X < 0) \leq \alpha\} \,.$$
2. *Any* eligible asset S is well behaved with respect to the Expected Shortfall based acceptance set
$$\mathscr{A}^\alpha = \{X \in \mathscr{X} \,;\, \mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta \leq 0\} \,.$$

Yet sometimes the only way to have finiteness for $\rho_{\mathscr{A},S}$ is that S_1 is bounded away from zero.

Example

An eligible asset S is well behaved with respect to the acceptance set $\mathscr{X}^+$ if and only if its payoff S_1 is bounded away from zero.

When is required capital a continuous function of financial positions?

Differently from the familiar case of cash-additive risk measures, the capital requirement $\rho_{\mathscr{A},S}$ may be *non continuous* even if it is finite for any position in $\mathscr{X}$.

Example

Let Ω be infinite and fix a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Consider an eligible asset with payoff S_1 such that $\mathbb{P}(S_1 < \varepsilon) > 0$ for all $\varepsilon > 0$. Then we can find $\alpha \in (0,1)$ so that the Value-at-Risk based required capital $\rho_{\mathscr{A}_\alpha,S} : \mathscr{X} \to \mathbb{R}$ is not continuous.

Indeed $\rho_{\mathscr{A}_\alpha,S}$ fails to satisfy the conditions given by the following theorem.

Theorem

Assume that the function $\rho_{\mathscr{A},S}$ is real valued. Then $\rho_{\mathscr{A},S}$ is continuous if and only if any of the following statements is true:

(i) $\operatorname{int}(\mathscr{A}) = \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) < 0\}$ *and* $\overline{\mathscr{A}} = \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) \leq 0\}$.

(ii) $\rho_{\operatorname{int}(\mathscr{A}),S} = \rho_{\mathscr{A},S} = \rho_{\overline{\mathscr{A}},S}$.

(iii) $\rho_{\mathscr{A},S}$ *is continuous from above and from below.*

Lipschitz continuity

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be an arbitrary acceptance set and assume that the payoff S_1 of the eligible asset is bounded away from zero. Then $\rho_{\mathscr{A},S}$ is everywhere finite and Lipschitz continuous, i.e. there exists $L > 0$ such that for all $X, Y \in \mathscr{X}$

$$\left|\rho_{\mathscr{A},S}(X) - \rho_{\mathscr{A},S}(Y)\right| \leq L \left\|X - Y\right\| \ .$$

Remark

Again the proposition applies in case S is the risk-free asset, Ω is finite or the σ-algebra $\mathcal{F}$ is finite.

The previous condition is sometimes but not always necessary.

Examples

1. Fix $\mathscr{A} := \mathscr{X}^+$. Then $\rho_{\mathscr{A},S}$ is everywhere finite and continuous if and only if S_1 is bounded away from zero.
2. Fix $\lambda \in \mathbb{R}$ and a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$, and consider the acceptance set $\mathscr{A} := \{X \in \mathscr{X} \,;\, \mathbb{E}_{\mathbb{P}}[X] \geq \lambda\}$. Then $\rho_{\mathscr{A},S}$ is everywhere finite and continuous for *any* choice of the eligible asset S.

Continuity in case of convex acceptance sets

The class of convex acceptance sets is special: if $\mathscr{A}$ is a convex acceptance set then any effective required capital $\rho_{\mathscr{A},S}$ is also robust.

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be a convex acceptance set and S an arbitrary eligible asset. Set $\mathrm{dom}(\rho_{\mathscr{A},S}) := \{X \in \mathscr{X}\,;\, \rho_{\mathscr{A},S}(X) \in \mathbb{R}\}$.

(i) *$\rho_{\mathscr{A},S}$ is Lipschitz continuous on a convenient neighborhood of each position $X \in \mathrm{int}(\mathrm{dom}(\rho_{\mathscr{A},S}))$.*

(ii) *If in particular $\rho_{\mathscr{A},S}$ is real valued, then $\rho_{\mathscr{A},S}$ is locally Lipschitz continuous, i.e. it is Lipschitz continuous on some neighborhood of each position $X \in \mathscr{X}$.*

Note that if $\mathscr{A}$ is a convex acceptance set and S is a generic eligible asset, then $\rho_{\mathscr{A},S}$ may be not effective, i.e. may be not everywhere finite, as already shown.

Example

Consider the convex (indeed coherent) acceptance set $\mathscr{A} := \mathscr{X}^+$. If the payoff S_1 is not bounded away from zero, then $\rho_{\mathscr{A},S}(-1_\Omega) = \infty$.

Continuity in case of coherent acceptance sets

In case of coherent acceptance sets we have a stronger result about robustness.

Proposition

If $\mathscr{A}$ is coherent and $\rho_{\mathscr{A},S}$ is real valued, then $\rho_{\mathscr{A},S}$ is Lipschitz continuous.

As a remarkable corollary, the capital requirement based on Expected Shortfall is a Lipschitz-continuous function of financial positions for *any* choice of the eligible asset.

Corollary

Consider the coherent acceptance set based on Expected Shortfall at level $\alpha \in (0,1)$

$$\mathscr{A}^{\alpha} = \{X \in \mathscr{X}\,;\, \mathrm{ES}_{\alpha}(X) = \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}_{\beta}(X)d\beta \leq 0\}\ .$$

Let S be an arbitrary eligible asset. Then $\rho_{\mathscr{A}^{\alpha},S}$ is everywhere finite, hence it is Lipschitz continuous.

Lower semicontinuity

We now focus on a weakened form of continuity, i.e. lower semicontinuity, which will be important while investigating the existence of an optimal eligible asset.

We recall that a function $f : \mathscr{X} \to [-\infty, \infty]$ is said to be *lower semicontinuous* if the set $\{X \in \mathscr{X} \,;\, f(X) \leq \lambda\}$ is closed for any $\lambda \in \mathbb{R}$.

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be an acceptance set and S an eligible asset. The following statements are equivalent.

(i) $\rho_{\mathscr{A},S}$ *is lower semicontinuous.*

(ii) $\overline{\mathscr{A}} = \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) \leq 0\}$.

(iii) $\rho_{\overline{\mathscr{A}},S} = \rho_{\mathscr{A},S}$.

(iv) $\rho_{\mathscr{A},S}$ *is continuous from above, i.e. if $X_n \downarrow X$ then $\rho(X_n) \uparrow \rho(X)$.*

Example

Consider $\mathscr{A}_\alpha = \{X \in \mathscr{X} \,;\, \mathrm{VaR}_\alpha(X) \leq 0\}$ and $\mathscr{A}^\alpha = \{X \in \mathscr{X} \,;\, \mathrm{ES}_\alpha(X) \leq 0\}$. Then both $\rho_{\mathscr{A}_\alpha,S}$ and $\rho_{\mathscr{A}^\alpha,S}$ are lower semicontinuous for any choice of the eligible asset S. This follows from the well-known continuity of VaR_α and ES_α.

Equality between risk measures

We assume that:

1. $\mathscr{A}$ and $\mathscr{B}$ are two acceptance sets;
2. S and R are two eligible assets with same price $S_0 = R_0 > 0$;
3. S and R are well behaved with respect to $\mathscr{A}$ and $\mathscr{B}$, respectively;
4. $\rho_{\mathscr{A},S}$ and $\rho_{\mathscr{B},R}$ are both lower semicontinuous.

Theorem

Let $\mathscr{S} = \{S, R\}$ and $m = \rho_{\mathscr{A},S}(0)$.

(i) $\rho_{\mathscr{A},S} = \rho_{\mathscr{B},R}$ *if and only if* $\overline{\mathscr{A}} = \overline{\mathscr{B}}$ *and* $\overline{\mathscr{A}} + \mathscr{M}_0(\mathscr{S}) = \overline{\mathscr{A}}$ *hold.*

(ii) *Assume that* $\mathscr{M}_m(\mathscr{S}) \cap \overline{\mathscr{A}} = \left\{ \frac{m}{S_0} S_1 \right\}$. *Then* $\rho_{\mathscr{A},S} = \rho_{\mathscr{B},R}$ *if and only if* $\overline{\mathscr{A}} = \overline{\mathscr{B}}$ *and* $S_1 = R_1$ *hold.*

Remark

We note that in case $\rho_{\mathscr{A},S}(0) = 0$, the condition $\mathscr{M}_0(\mathscr{S}) \cap \overline{\mathscr{A}} = \{0\}$ implies the non-acceptability of *fully-leveraged* positions in S and R, i.e. portfolios in S and R with zero initial value.

Does an optimal eligible asset exist?

Given a fixed acceptance set $\mathscr{A}$ and an eligible asset S, we ask if it is possible that for any position $X \in \mathscr{X}$

$$\rho_{\mathscr{A},S}(X) \leq \rho_{\mathscr{A},R}(X) \quad \text{for any eligible asset } R \text{ different from } S \ ,$$

and for at least one position $Y \in \mathscr{X}$ we indeed have

$$\rho_{\mathscr{A},S}(Y) < \rho_{\mathscr{A},R}(Y) \ .$$

We will assume here that:

1. S and R are two eligible assets with same price $S_0 = R_0 > 0$;
2. S and R are both well behaved with respect to a fixed acceptance set $\mathscr{A} \subset \mathscr{X}$;
3. $\rho_{\mathscr{A},S}$ and $\rho_{\mathscr{A},R}$ are both lower semicontinuous.

Example

As mentioned above, if $\mathscr{A}$ is the acceptance set based on Value-at-Risk or on Expected Shortfall such assumptions are satisfied by any eligible assets S and R with same initial price.

There is no optimal eligible asset

Theorem

Under the above assumptions, the following statements hold.

(i) *If* $\rho_{\mathscr{A},S}(X) \leq \rho_{\mathscr{A},R}(X)$ *for every* $X \in \mathscr{X}$ *then*

$$\rho_{\mathscr{A},S} = \rho_{\mathscr{A},R} \ .$$

(ii) *Assume that* $\mathscr{S} = \{S, R\}$ *and* $\mathscr{M}_{\rho_{\mathscr{A},S}(0)}(\mathscr{S}) \cap \overline{\mathscr{A}} = \left\{ \frac{\rho_{\mathscr{A},S}(0)}{S_0} S_1 \right\}$.
If $\rho_{\mathscr{A},S}(X) \leq \rho_{\mathscr{A},R}(X)$ *for every* $X \in \mathscr{X}$ *then*

$$S_1 = R_1 \ .$$

Remark

- Given an eligible asset, we cannot find any other eligible asset that will be *strictly more efficient*.
- Fix an eligible asset R. If fully-leveraged positions are not acceptable, there does not exist any eligible asset different from R which is just *more efficient* than R.

Eligible asset and numeraire asset

In Filipović (2008) a different sort of optimality is addressed.
The author investigates the impact a change in numeraire has on a fixed *convex* risk measure $\rho : \mathscr{X} \to \mathbb{R}$.
A key consideration in this treatment is that a numeraire U induces an acceptance set

$$\mathscr{A}^U := \left\{ X \in \mathscr{X} \, ; \, \rho\left(\frac{X}{U}\right) \leq 0 \right\} \, .$$

The central result is that *no optimal numeraire exists*: there does not exist a numeraire U leading to lower capital requirements than those associated to any other numeraire, i.e. such that $\mathscr{A}^V \subset \mathscr{A}^U$ for any other numeraire V.

With respect to our treatment, we point out the following remarks.

- First, note again that dealing with *discounted* positions forces the numeraire to be bounded away from zero in order to guarantee that discounted positions still belong to $\mathscr{X}$.
- Second, it is unclear how to interpret this non-optimality result from an economic perspective: if acceptance is meant to reflect the adequate capitalization of a financial institution, it should not depend on the particular numeraire we have chosen to account in.

Capital requirements with respect to several eligible assets

We focus back on the setting where required capital is determined in the context of more than one single eligible asset.

We recall that the required capital $\rho_{\mathscr{A},\mathscr{S}} : \mathscr{X} \to [-\infty, \infty]$ is defined by

$$\begin{aligned}\rho_{\mathscr{A},\mathscr{S}}(X) &:= \inf\{\pi(Z) \in \mathbb{R}\,;\, Z \in \mathscr{M}(\mathscr{S}) \,:\, X + Z \in \mathscr{A}\} \\ &= \inf\{m \in \mathbb{R}\,;\, Z \in \mathscr{M}_m(\mathscr{S}) \,:\, X + Z \in \mathscr{A}\}\end{aligned}$$

where

- $\mathscr{A} \subset \mathscr{X}$ is a fixed acceptance set,
- $\mathscr{S} = \{S^1, \cdots, S^N\}$ is a given class of eligible assets made of tradable assets S^j with initial price $S_0^j > 0$ and payoff $S_1^j \in \mathscr{X}$ such that $S_1^j(\omega) > 0$ for any $\omega \in \Omega$,
- $\mathscr{M}(\mathscr{S}) = \mathrm{span}(\{S_1^1, \cdots, S_1^N\})$ is the corresponding arbitrage-free marketed space,
- $\pi : \mathscr{M}(\mathscr{S}) \to \mathbb{R}$ is the linear pricing functional on $\mathscr{M}(\mathscr{S})$,
- $\mathscr{M}_m(\mathscr{S}) := \{Z \in \mathscr{M}(\mathscr{S})\,;\, \pi(Z) = m\}$.

From several eligible assets to a single one

We start by showing that, surprisingly enough, the study of capital requirements with respect to a class of eligible assets can be reduced to the single-asset case.

Theorem

Let $\mathscr{A}$ be an arbitrary subset of $\mathscr{X}$. Assume $\mathscr{T}$ is a non-empty subset of $\mathscr{S}$. Then

$$\rho_{\mathscr{A},\mathscr{S}} = \rho_{\mathscr{A}+\mathscr{M}_0(\mathscr{S}),\mathscr{T}} \ .$$

In particular for any $S \in \mathscr{S}$ we have

$$\rho_{\mathscr{A},\mathscr{S}} = \rho_{\mathscr{A}+\mathscr{M}_0(\mathscr{S}),S} \ .$$

Remark

- By properly enlarging the set $\mathscr{A}$, we can always express the capital requirement $\rho_{\mathscr{A},\mathscr{S}}$ with respect to the class $\mathscr{S}$ in terms of a capital requirement with respect to any particular asset S belonging to the class $\mathscr{S}$ itself.
- If $\mathscr{A}$ is an acceptance set and $\mathscr{A}+\mathscr{M}_0(\mathscr{S}) \neq \mathscr{X}$, then $\mathscr{A}+\mathscr{M}_0(\mathscr{S})$ is also an acceptance set.

Absence of acceptability arbitrage

The condition $\mathscr{A} + \mathscr{M}_0(\mathscr{S}) \neq \mathscr{X}$ has a clear economic interpretation.

- If it does not hold then any financial position can be made acceptable at zero cost, by hedging with portfolios long and short of eligible assets, a situation that would cast doubts on the effectiveness of the choice of the acceptance set $\mathscr{A}$.

We are therefore lead to set the following assumption.

Assumption

We will say that the assumption $NAA(\mathscr{A}, \mathscr{S})$ of *absence of acceptability arbitrage* holds if $\mathscr{A} + \mathscr{M}_0(\mathscr{S}) \neq \mathscr{X}$.

The following result shows that absence of acceptability arbitrage is a sound assumption: under $NAA(\mathscr{A}, \mathscr{S})$ there does not exist a price $m \in \mathbb{R}$, in particular a negative price $m < 0$, such that we can make every financial position acceptable at cost m.

Proposition

If $\mathscr{A} + \mathscr{M}_{m_0}(\mathscr{S}) \neq \mathscr{X}$ for some $m_0 \in \mathbb{R}$, then $\mathscr{A} + \mathscr{M}_m(\mathscr{S}) \neq \mathscr{X}$ for all $m \in \mathbb{R}$.

Sufficient conditions for $NAA(\mathscr{A},\mathscr{S})$

We start by noting that given an acceptance set $\mathscr{A} \subset \mathscr{X}$ we obviously have $\mathscr{A} \subseteq \mathscr{A} + \mathscr{M}_0(\mathscr{S}) \subseteq \mathscr{X}$.

Proposition

If $\mathscr{A} = \mathscr{A} + \mathscr{M}_0(\mathscr{S})$ then $NAA(\mathscr{A},\mathscr{S})$ holds. Moreover the following statements are equivalent:

(i) $\rho_{\mathscr{A},\mathscr{S}}$ *is everywhere finite;*

(ii) *there exists $S \in \mathscr{S}$ such that $\rho_{\mathscr{A},S}$ is everywhere finite.*

Proposition

Assume that one of the following equivalent statements is fulfilled:

(i) $\mathrm{int}(\mathscr{A}) \cap \mathscr{M}_m(\mathscr{S}) = \emptyset$ *for some $m \in \mathbb{R}$;*

(ii) $\mathrm{int}(\mathscr{A}) \cap \mathscr{M}_m(\mathscr{S}) = \emptyset$ *for some $m \leq 0$.*

Then $NAA(\mathscr{A},\mathscr{S})$ holds true.

The second sufficient condition becomes also necessary in case one additional assumption is made.

The case of a portfolio bounded away from zero

Assumption

We will assume that the class $\mathscr{S}$ of eligible assets allows for a portfolio $Z^* \in \mathscr{M}(\mathscr{S})$ which is *bounded away from zero*. This happens in particular if one of the assets belonging to $\mathscr{S}$ is itself bounded away from zero, for example it is the risk-free asset.

Remark

This assumption is meaningful from an economic point of view.

- Adding one more asset to a given class of eligible assets allows for a *lower required capital* for each financial position.
- The point is to guarantee that the new required capital is not *too low*, i.e. equal to $-\infty$.

Characterization of $NAA(\mathscr{A},\mathscr{S})$

The following proposition displays sufficient and necessary conditions so that the assumption $NAA(\mathscr{A},\mathscr{S})$ is fulfilled.

Proposition

Assume there exists a portfolio $Z^* \in \mathscr{M}(\mathscr{S})$ which is bounded away from zero. Then $\rho_{\mathscr{A},\mathscr{S}}(X) < \infty$ for any $X \in \mathscr{X}$.

Moreover $NAA(\mathscr{A},\mathscr{S})$ is equivalent to any of the following statements:

(i) $\rho_{\mathscr{A},\mathscr{S}}(X) > -\infty$ *for all* $X \in \mathscr{X}$;

(ii) $\rho_{\mathscr{A},\mathscr{S}}(X) > -\infty$ *for some* $X \in \mathscr{X}$;

(iii) $\mathrm{int}(\mathscr{A}) \cap \mathscr{M}_m(\mathscr{S}) = \emptyset$ *for some* $m \in \mathbb{R}$;

(iv) $\mathrm{int}(\mathscr{A}) \cap \mathscr{M}_m(\mathscr{S}) = \emptyset$ *for some* $m \leq 0$.

Acceptance sets that are cones

Proposition

Consider an acceptance set $\mathscr{A} \subset \mathscr{X}$ that is a cone. Assume that $\mathscr{M}(\mathscr{S})$ contains a portfolio that is bounded away from zero and that all the eligible assets have price 1. Then the following conditions are equivalent:

(i) *$NAA(\mathscr{A}, \mathscr{S})$ holds;*

(ii) $\mathrm{int}(\mathscr{A}) \cap \mathscr{M}_0(\mathscr{S}) = \emptyset$.

Corollary

Consider an acceptance set $\mathscr{A} \subset \mathscr{X}$ that is a cone. Assume that the payoff S_1^1 of the first eligible asset is bounded away from zero. Then the following statements are equivalent:

(i) *$NAA(\mathscr{A}, \mathscr{S})$ holds;*

(ii) *for any $\lambda_2, \cdots, \lambda_N \in \mathbb{R}$ we have*

$$\rho_{\mathscr{A},S^1}\left(\sum_{j=2}^{N} \lambda_j (S_1^j - S_1^1)\right) \geq 0 \ .$$

Value-at-Risk and Expected Shortfall

Consider the Value-at-Risk based acceptance set at level $\alpha \in (0,1)$

$$\mathscr{A}_\alpha = \{X \in \mathscr{X}\,;\, \mathrm{VaR}_\alpha(X) \le 0\}\ .$$

Assume that $S_1^1 = 1_\Omega$ and that all the eligible assets have price 1. Then the following statements are equivalent:

(i) $NAA(\mathscr{A}_\alpha, \mathscr{S})$ holds;

(ii) for any $\lambda_2, \cdots, \lambda_N \in \mathbb{R}$ we have

$$\mathrm{VaR}_\alpha\left(\sum_{j=2}^{N} \lambda_j (S_1^j - 1_\Omega)\right) \ge 0\ .$$

If $\mathscr{S}$ is made of just two assets, namely the risk-free asset and another asset S, the previous equivalence becomes:

(i) $NAA(\mathscr{A}_\alpha, \mathscr{S})$ holds;

(ii) $\mathrm{VaR}_\alpha(S_1 - 1_\Omega) \ge 0$ and $\mathrm{VaR}_\alpha(1_\Omega - S_1) \ge 0$.

The result remains true with ES_α instead of VaR_α.

Existence of acceptability arbitrage: the VaR and ES case

Example

Take $\mathscr{S}$ consisting of the risk-free asset and of an eligible asset S such that for a convenient $\varepsilon > 1$

$$\mathbb{P}(S_1 \geq \varepsilon) = 1 \ .$$

Since

$$\mathbb{P}(S_1 - 1_\Omega < \varepsilon - 1) = 0 \ ,$$

for any $\alpha \in (0,1)$ we have

$$\mathrm{VaR}_\alpha(S_1 - 1_\Omega) < 0 \ .$$

Hence *NAA*$(\mathscr{A}_\alpha, \mathscr{S})$ does not hold for *any* $\alpha \in (0,1)$.

Recall that for $X \in \mathscr{X}$ and $\alpha \in (0,1)$ we have set

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta \ .$$

As a simple consequence we obtain that *NAA*$(\mathscr{A}^\alpha, \mathscr{S})$ does not hold for *any* $\alpha \in (0,1)$ as well.

Further examples

Simple-scenario acceptance sets

Fix $\omega_0 \in \Omega$ and $\lambda \in \mathbb{R}$, and consider the acceptance set

$$\mathscr{A} := \{X \in \mathscr{X}\ ;\ X(\omega_0) \geq \lambda\}\ .$$

The following statements are equivalent:

(i) $NAA(\mathscr{A}, \mathscr{S})$ holds;

(ii) $S_1^1(\omega_0) = \cdots = S_1^N(\omega_0)$.

Expected-value based acceptance sets

Fix a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and $\lambda \in \mathbb{R}$. Consider the acceptance set

$$\mathscr{A} := \{X \in \mathscr{X}\ ;\ \mathbb{E}_{\mathbb{P}}(X) \geq \lambda\}\ .$$

The following statements are equivalent:

(i) $NAA(\mathscr{A}, \mathscr{S})$ holds;

(ii) $\mathbb{E}_{\mathbb{P}}\left(\frac{S_1^1}{S_0^1}\right) = \cdots = \mathbb{E}_{\mathbb{P}}\left(\frac{S_1^N}{S_0^N}\right)$;

(iii) $\mathscr{A} + \mathscr{M}_0(\mathscr{S}) = \mathscr{A}$.

Further examples

No-arbitrage and acceptability arbitrage

There is a class of acceptance sets $\mathscr{A}$ for which $NAA(\mathscr{A},\mathscr{S})$ holds regardless of the choice of the eligible assets.

Fix a position $U \in \mathscr{X}$ and consider an acceptance set $\mathscr{A} \subset \mathscr{X}$ such that

$$\mathscr{A} \subseteq \{X \in \mathscr{X}\,;\, X \geq U\}\ .$$

Then $NAA(\mathscr{A},\mathscr{S})$ holds for *any* choice of the class $\mathscr{S}$ (satisfying no-arbitrage). Indeed set $X := U - 1_\Omega$. Then $X \notin \mathscr{A} + \mathscr{M}_0(\mathscr{S})$, since otherwise there would exist $Z \in \mathscr{M}_0(\mathscr{S})$ such that $Z \geq 1_\Omega$, contradicting no-arbitrage.

References

- T. Arai, Convex risk measures on Orlicz spaces: inf-convolution and shortfall, *Mathematics and Financial Economics*, 3: 73-88, 2010.
- Ph. Artzner, F. Delbaen, J.-M. Eber and D. Heath, Coherent measures of risk, *Mathematical Finance*, 9: 203-228, 1999.
- S. Biagini and M. Frittelli, A unified framework for utility maximization problems: an Orlicz space approach, *Annals of Applied Probability*, 18: 929-966, 2008.
- S. Biagini and M. Frittelli, On the extension of the Namioka-Klee theorem and on the Fatou property for risk measures, in *Optimality and risk: modern trends in Mathematical Finance*, Springer, 1-28, 2009.
- P. Cheridito and T. Li, Risk measures on Orlicz hearts, *Mathematical Finance*, 19: 184-214, 2009.
- D. Filipović and M. Kupper, Monotone and cash-invariant convex functions and hulls, *Insurance: Mathematics and Economics*, 41: 1-16, 2007.
- D. Filipović, Optimal numeraires for risk measures, *Mathematical Finance*, 18: 333-336, 2008.
- H. Föllmer and A. Schied, Convex measures of risk and trading constraints, *Finance and Stochastics*, 6: 429-447, 2002.

H. Föllmer and A. Schied, *Stochastic finance: an introduction in discrete time*, De Gruyter, 3rd edition, 2011.

M. Frittelli and E. Rosazza Gianin, Putting order in risk measures, *Journal of Banking & Finance*, 26: 1473-1486, 2002.

M. Frittelli and G. Scandolo, Risk measures and capital requirements for processes, *Mathematical Finance*, 16: 589-612, 2006.

M. Kaina and L. Rüschendorf, On convex risk measures on L^p-spaces, *Mathematical Methods of Operations Research*, 69: 475-495, 2009.

D.G. Konstantinides and C.E. Kountzakis, Risk measures in ordered normed linear spaces with non-empty cone-interior, *Insurance: Mathematics and Economics*, 48: 111-122, 2011.

C.E. Kountzakis, Risk measures on ordered non-reflexive Banach spaces, *Journal of Mathematical Analysis and Applications*, 373: 548-562, 2011.

Contacts

For any further question and comment please refer to:

- Walter Farkas
  walter.farkas@bf.uzh.ch
- Pablo Koch-Medina
  Pablo_KochMedina@swissre.com
- Cosimo-Andrea Munari
  munari@math.ethz.ch