Social learning by chit-chat

Edoardo Gallo*

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Abstract

Individuals learn by chit-chatting with colleagues, friends and strangers as a by-product of their daily activities. We formulate a novel framework to investigate how the speed of learning by chit-chat depends on the structure of the environment. A network represents the environment that individuals navigate to interact with each other. We derive an exact formula to compute how the expected time between meetings depends on the underlying network structure and we use this quantity to investigate the speed of learning in the society. Comparative statics show that the speed of learning is sensitive to a mean-preserving spread of the degree distribution (MPS). Specifically, if the number of individuals is low (high), then a MPS of the network increases (decreases) the speed of learning. The speed of learning is the same for all regular networks independent of network connectivity. An extension explores the effectiveness of one agent, the influencer, at influencing the learning process.

Keywords: social learning, network, speed of learning, mean preserving spread, influencer.

JEL: D83, D85.

*Address: University of Oxford, Nuffield College, New Road, Oxford OX1 1NF, UK. Email: edoardo.gallo@economics.ox.ac.uk. I am especially grateful to Meg Meyer for her invaluable help and guidance throughout this project. Many thanks to Matt Elliott, Sanjeev Goyal, Ben Golub, Matt Jackson, Manuel Mueller-Frank, Adam Szeidl, and Peyton Young for helpful comments and suggestions. Thanks to seminar participants at Stanford University, the University of Oxford, the University of Florence, Middlesex University and Nanyang Technological University (Singapore); and to conference participants at the 4th International Conference on Game Theory and Management (St. Petersburg), the ESF Workshop on Information and Behavior in Networks (Oxford) and the Theory Transatlantic Workshop (Paris).
The less intimate interactions that we have ignored - discussions over backyard fences, casual encounters while taking walks, or standing in line at the grocery store, and so on - may be politically influential even though they do not occur between intimate associates.

Huckfeldt [1986]

1 Introduction

In 2009 Air New Zealand (ANZ) paid $777 to 30 volunteers who agreed to shave their head in order to wear a temporary tattoo saying “Need A Change? Head Down to New Zealand. www.airnewzealand.com.” Why? ANZ’s Director of Marketing explained that half of the participants were kiwi expats and half had visited New Zealand recently, so they were “ideal brand ambassadors: when co-workers or strangers behind them in the grocery store line asked about New Zealand, they could speak enthusiastically right off the top of their heads.”

The objective of this paper is to formulate a model of the learning process typified by this example, which we dub social learning by chit-chat, and to investigate the speed of learning in this framework. This learning process is distinctly different from the standard learning models in the economic literature, in which agents purposely learn how to play an underlying game. In the ANZ example the agents are strategic as they go about their daily activities, but these activities are orthogonal to the learning process. The presumption of the social learning literature is that individuals actively learn from the decisions and experiences of their neighbors and/or social relations, while in the ANZ example an individual can passively learn from another individual independently of the presence of a social relation.

The main determinants of the speed of learning in the social learning by chit-chat framework are the size of the population and the environment these individuals are living in. For instance, if we think of the environment as a physical environment, the speed of learning may be very different in a scarcely populated village where there is one central square vis-à-vis a metropolis buzzing with people with multiple meeting points. Moreover, in this framework individuals are learning as a by-product of their other activities and they do not rely exclusively on their social relations to learn: they discuss politics with their work colleagues during coffee break, they chat about fashion trends with strangers at the gym, or they exchange opinions on the latest movie with acquaintances at the golf course.

We use environment as a broad umbrella term that encompasses a variety of meanings. The most proximate meaning, which we will mostly refer to for expository purposes, is physical environments such as the various locations in an urban area or the meeting places within an organization. Another meaning is virtual environments: online message boards, forums and blogs are the sites where individuals exchange information while they navigate through them.

the web. Finally, the term environment may also be interpreted as an *abstract space of circumstances* that prompt individuals to exchange views on a topic. An example could be the set of topics that may be salient in an individual’s mind: if the same topic is salient in two individuals then they are in the same location in this abstract space and they may exchange information. We represent the structure of the environment using a network.

Specifically, we consider a population of \( n \) agents who are random walkers on a given network \( g \). At each point in time a randomly chosen individual moves from her current site to one of the neighboring sites, with equal probability for each neighboring site. If there are other individuals on the destination site then one of them is randomly chosen and a meeting occurs. Otherwise, no meeting takes place. Within this framework we model learning by chit-chat as follows. At time \( T \) each agent receives a noisy, independent and identically distributed signal about the true state of the world. Every time two agents meet they truthfully reveal their current beliefs and they update their own belief by taking a weighted average of their previous belief and the belief of the agent that they have met, where the weight given to the other agent’s belief is (weakly) less than the weight given to their own belief.

In the long term all agents’ beliefs converge (almost surely) to the same value, and the distribution of the limit belief has a mean equal to the average of the initial signals. Theorem 1 presents the first main result of the paper: the derivation of an explicit expression for the *expected time* \( \tau \) an individual waits between two meetings as a function of the number of individuals and the network structure. The formula for \( \tau \) is an exact result and the comparative statics in the paper are formally stated in terms of how \( \tau \) varies with changes in the network. We interpret results on the variations in \( \tau \) as variations in the speed of learning by using a mean-field approximation.

The second main result, in theorem 2, relates the expected time between meetings, and therefore the speed of learning, to the underlying network structure. *If the number of agents is low relative to the number of sites, then a mean preserving spread of the degree distribution decreases the expected time between meetings*, i.e. the speed of learning increases with the level of variability in number of connections across sites. The intuition is that in a heterogenous environment most of the paths in the network lead to the well-connected sites, and the presence of these sites increases the probability of meeting between the small number of agents. On the other hand, *if the number of agents is high relative to the number of sites, then a mean preserving spread of the degree distribution increases the expected time between meetings*. The intuition is that in a homogenous environment the sites have a similar number of connections so the agents will be distributed uniformly across them, and this minimizes the probability that one agent moves to an unoccupied site.

One may conjecture that the density of connections in the network would also have an effect on the speed of learning. However, proposition 1 shows that *the expected time between meetings is invariant to changes in connectivity of the environment*. Specifically,
it shows that the speed of learning is the same for any environment that is represented by a regular network independent of network connectivity. The intuition is that there are two effects at play that cancel each other. On the one hand an increase in connectivity allows agents access to a larger number of sites from their current location, increasing in this way the probability of access to an occupied site where a meeting can occur. On the other hand this increase in accessibility means that if two agents are at neighboring sites then it is less likely that one of them will randomly move to the other’s location, so this effect decreases the frequency of meetings in better connected environments. These two effects exactly cancel each other in regular networks, so the expected time between meetings, and therefore the speed of learning, is invariant to changes in the connectivity of the network.

ANZ’s marketing strategy was a passive way to exploit learning by chit-chat: the volunteers had no vested interest and they were not instructed on how to react to others’ questions about their distinctive tattoo. However, the importance of this learning process makes it a tempting channel to influence public opinion. For instance, political campaigns regularly recruit volunteers to station at specific locations in order to sway undecided voters through word-of-mouth.\(^2\)

The last part of the paper extends the basic framework to analyze the impact that one agent, the *influencer*, has on the learning by chit-chat process. Specifically, the goal is to highlight how the structure of the environment makes a society more or less susceptible to the presence of the influencer. We model the influencer as an agent who does not change her belief and who is positioned at a site in the network, instead of traveling around for purposes orthogonal to the learning process.

The first result is that the site that maximizes the effectiveness of the influencer is the most connected site in the network. Assuming that the influencer is at the most connected site, a society is minimally susceptible to the presence of the influencer if and only if the underlying environment is a regular network. Moreover, if the number of individuals in the society is above a threshold, then a mean preserving spread of the degree distribution of the underlying network makes a society more susceptible to the presence of the influencer. Two effects drive this result. The first one is that a direct application of theorem 2 shows that in a more heterogeneous environment the frequency of meetings involving non-influencers decreases. The second one is that a shift to a more heterogeneous environment leads to a (weak) increase in the connectivity of the most connected site where the influencer resides, leading to an increase in the frequency of meetings involving the influencer. Thus, the overall impact of the influencer on the learning process increases.

The rest of this section presents a literature review. Section 2 describes the general framework and the learning process. Section 3 considers some stylized networks to illustrate the intuition behind the main results, which are presented in section 4. Section 5

\(^2\)The resources spent on this type of campaigning have significantly increased in recent years, see, e.g., Panagopoulous and Weilhouwer [2008].
extends the basic model to investigate the effect of an influencer. Appendix A contains all the proofs.

1.1 Literature review

The literature on learning in economics is extensive and the purpose here is not to give a complete overview of this large body of work. This section will focus on social learning models with a network to highlight how this paper differs from previous contributions.

Bala and Goyal [1998] examine an observational social learning model in which agents take actions at regular intervals and these actions generate noisy payoffs. Agents learn from the payoffs they receive and from observing the payoffs of their neighbors in the social network. This type of social learning is called observational in the sense that agents observe the actions, not the beliefs, of the other agents. The main result is that everyone converges to the optimal action as long as there isn’t a subset of agents that are too ‘influential’ in terms of their position in the network. Recent contributions by Acemoglu et al. [2012] and Lamberson [2010] present further results in this framework.

DeMarzo et al. [2003] investigate a different set-up in which agents receive a noisy signal of the underlying state of the world at time 0 and then they update this signal by taking a weighted average of the beliefs of their neighbors in the social network. The weights are fixed by the exogenously given network. Agents report their beliefs truthfully, and they are boundedly rational because they do not take into account repetitions of information in the social network. They show that if there is sufficient communication then the beliefs of all agents converge in the long-run to a weighted average of initial beliefs, but the information is not aggregated optimally unless an agent’s prominence in the social network coincides with the accuracy of his initial signal. They also show that the speed of convergence is governed by the size of the second eigenvalue of the network matrix. Golub and Jackson [2010] generalize some of these results.

The main conceptual difference between learning by chit-chat and these previous contributions is the function played by the network. In social learning models the network constrains whom the agents can learn from, either because of physical or social constraints. The interpretation of the physical constraints is that agents have restrictions on their mobility that do not allow them to observe all other agents in the society. A plausible interpretation of the social constraints is that agents rely only upon agents they trust in order to learn about payoff-relevant information. On the other hand, in learning by chit-chat the function of the network is to model the structure of the environment where agents live and that determines the frequency of their meetings to exchange information. The presence of a social tie is not a prerequisite for exchanging information because agents are learning while interacting with a variety of individuals as they are going about their daily activities. Moreover, the type of information being exchanged is not particularly

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3This learning rule was first proposed in DeGroot [1974].
sensitive so agents learn from friends as well as from strangers. Examples include learning about the best restaurant in town, fashion trends or the latest movie release.

This major conceptual difference motivates the use of a novel methodology to investigate the speed of convergence in the learning by chit-chat context. We rely on a result by Kac [1947] in the theory of stochastic processes to explicitly compute how the expected time an agent waits between two interactions depends on the underlying network structure. This explicit solution permits a very clear comparative statics analysis that shows which type of changes in network structure have an impact on the speed of convergence, and how that depends on the number of agents in the society. This is in sharp contrast to the notorious difficulty in deriving results relating the speed of convergence to the social network structure in the social learning literature.

An important question in the social learning literature is whether the society aggregates information efficiently and converges to the payoff-maximizing outcome. In observational social learning models the spirit of the results is that there is convergence to the payoff-maximizing outcome as long as some very weak conditions on the network structure are satisfied. The usual form of these conditions is that there should not be one or a few agents that have a disproportionate influence on the rest of society. On the other hand, in the DeMarzo et al. [2003] framework information is usually not aggregated optimally because agents’ influence differ depending on their position in the network and their influence does not usually correlate with the informativeness of their initial signal. DeMarzo et al. [2003] show how to compute the influence of each agent on the common limiting belief.

The learning by chit-chat framework is similar to the DeMarzo et al. [2003] model because learning is not observational and upon meeting agents truthfully reveal their beliefs about the underlying state of the world. However, in the basic learning by chit-chat framework the agents are interchangeable, so in the long-term each agent has the same influence on the common belief the society converges to. Our extension involving an influencer can be viewed as complementary to the approach in DeMarzo et al. [2003] because we show how the effectiveness of the influencer depends on the environment.

Finally, it is important to notice that the main results of the paper are applicable beyond the context of the learning model analyzed in this paper. The general framework is a stochastic dynamic system of agents travelling on a network and the main results relate the structure of the network with the frequency of encounters. This process is potentially relevant to other economic phenomena such as the impact of epidemics and the spread of innovations, but this paper will focus on the application to learning.

2 The Model

This section presents the main elements of the model: the network concepts and terminology, the way agents move on the network and the learning process.
Network. Consider a finite set of nodes $S = \{1, \ldots, s\}$, which will be called sites. The links among the sites are described by an adjacency matrix $g = [g_{ij}]$, where $g_{ij} \in \{0, 1\}$ and $g_{ij} = g_{ji}$ for all $i, j \in N$ ($j \neq i$) and $g_{ii} = 0$ and for all $i$. Let $N_a(g) = \{b \in S | g_{ab} = 1\}$ be the neighborhood of site $a$ in network $g$. The degree $d_a(g) = |N_a(g)|$ is the size of the neighborhood of site $a$ in network $g$, i.e., the number of sites directly connected to site $a$. Let $D(g) = \sum_i d_i(g)$ be the sum of the degrees of all the sites in the network, which is equal to twice the number of links in the network.

Denote by $P(d)$ the degree distribution of sites in a network, and let $\mu[P(d)]$ denote the mean of the distribution. The degree distribution is a description of the relative frequencies of sites that have different degrees. The comparative statics analysis will investigate changes in the network structure that are captured by a mean-preserving spread of this distribution. The following is a more formal definition of this notion.

**Definition 1.** A distribution $P'(d)$ is a mean-preserving spread (MPS) of another distribution $P(d)$ if $\mu[P(d)] = \mu[P'(d)]$ and if $\sum_{Y=0}^{\infty} \left[ \sum_{d=0}^{Y} P'(d) - P(d) \right] \geq 0$ for all $Z \in [1, s]$.

Consider two networks $g$ and $g'$ with degree distributions $P(d)$ and $P'(d)$ respectively. If $P'(d)$ MPS $P(d)$ then the $g'$ network is more heterogeneous than the $g$ network, where more heterogeneous means that there is more variability across sites in $g'$ than across sites in $g$ in terms of the number of their connections. In the rest of the paper we will also use the shorthand terminology that a network $g'$ is a mean-preserving spread of $g$ to mean that the degree distribution of the network $g'$ is a mean-preserving spread of the degree distribution of the network $g$.

Process. Consider a discrete-time process $t \in \{0, 1, 2, \ldots\}$. Let $N \in \{1, \ldots, n\}$ be a finite set of agents that travel on the network $g$. At time $t = 0$ agents are randomly allocated to sites. At each $t$ one agent $i \in N$ is randomly picked and he moves from his current site $a \in S$ to a new site $b \in N_a(g)$. Note that all agents are equally likely to be picked regardless of their current location and the previous history. The agents are random walkers so there is an equal probability that an agent moves to any of the neighboring sites $b \in N_a(g)$. Once agent $i$ arrives at site $b$ a meeting may occur. If there are $m \in (0, n)$ agents at site $b$ then one agent $j$ is randomly picked to meet $i$. If there are no agents on site $b$ then no meeting occurs.

The state of the system is a vector $x_{ij}^t \in X \in \mathbb{R}^{n+2}$ that captures the position of each agent in the network at time $t$ after agent $i$ has moved, and the subscripts $i$ and $j$ denote the agents that have been selected to interact at $t$, with $j = 0$ if $i$ has moved to an empty site. Note that the definition of a state explicitly includes the information on whether an interaction occurred and between which agents.

Learning. At time $T$, where $T$ is large, each agent receives a signal $\theta_i = \theta + \epsilon_i$ about an
unknown state of the world whose true value is $\theta$. Assume that $\epsilon_i \sim N(0, \sigma)$ is an error term independently drawn for each $i$ from a common normal distribution with mean 0. The learning process is as follows. Suppose that at time $t > T$ agent $i$ moves to site $b$ and agent $j$, who was already at $b$, is picked to meet with $i$. During the meeting $i$ and $j$ truthfully reveal their current view on the underlying state of the world, and they update their own view accordingly. The initial signal is never revealed.

More formally, let $x_i^t$ denote agent $i$’s beliefs at time $t$. At $t = T$ we assume that $x_i^T = \theta_i = \theta + \epsilon_i$. Suppose that at time $t > T$ agent $i$ meets with agent $j$, then after the meeting agent $i$’s revised beliefs are $x_i^t = x_i^{t-1} + \alpha(x_j^{t-1} - x_i^{t-1})$ where $\alpha \in (0, \frac{1}{2})$ is an exogenous parameter which is the same for all agents.\(^4\) The updating rule is the same one used in DeMarzo et al. [2003] and it is very simple: the agent’s belief after the meeting is a weighted average of his previous belief and the previous belief of the agent he has met, where the factor $\alpha$ captures the weight given to the other agent’s belief.

The model makes the following two assumptions on the interaction process: (a) interactions can occur only as a result of movement by one of the agents; (b) one and only one interaction is possible per time period. The primary purpose of these two assumptions is to realistically capture the social learning by chit-chat process described in the introduction. Let’s imagine a prototypical story of learning by chit-chat: an individual $i$ comes to a grocery store and starts a conversation with $j$, as they are busy with their one-to-one conversation other individuals may move around and start further one-to-one conversations, until at a certain point the conversation between $i$ and $j$ stops and $i$ (stays idle for a while until he) meets another individual who just moved to the location, etc. Assumption (a) captures the reality that learning happens because the social mix at any given location is changing as a result of individuals moving in the environment. Assumption (b) captures the fact that learning is a result of personal interaction, not public broadcasting to everyone that happens to be at that location.

The secondary purpose of these assumptions is simplicity and analytical tractability. An extension to (a) would be to allow agents to stay put with some probability when they are picked to move. Similarly, one could relax assumption (b) to allow interactions between more than two agents. Both these extension would fit the prototypical story and the learning by chit-chat framework. There are different potential formulations of each extension, which would introduce additional complexity to the derivation of the formula in theorem 1. The two effects driving the comparative statics in theorem 2 would stay unchanged, but each extension would introduce one additional effect, whose strength would depend on the exact formulation of the extension. Given the novelty of the framework in this paper, we decided to prioritize simplicity and avoid the risk of

\(^4\)The restriction that $\alpha < \frac{1}{2}$ is a sufficient, but not necessary, condition to ensure convergence. Given that our main focus is not the learning rule we chose a more restrictive bound to avoid dealing with convergence issues as $\alpha$ approaches 1.
obfuscating the intuition behind the main results. The exploration of these extensions is therefore left to future work.

3 Motivating examples

The general framework allows for any network to represent the environment that agents navigate. This generality may sometimes obfuscate the intuition behind the results so here we consider a class of simple stylized networks for illustrative purposes.

Let \( s \) denote the number of sites and consider a class of undirected, unweighted network architectures that we will label two-layer star networks. A two-layer star network has a central node linked with \( s - k - 1 \) nodes that form the inner layer. Each node in the inner layer is then linked to at most one of the \( k \) nodes in the outer layer, where \( k \leq \frac{1}{2}(s - 1) \).

Denote by \( g^*_k \) a two-layer star network with \( k \) nodes in the outer layer. Clearly, if \( k = 0 \) then we have the star network which we denote simply by \( g^* \). Figure 1 shows all the elements of the class of two-layer star networks for \( s = 7 \). Proceeding clockwise from the top left corner we have the two-layer star networks \( g^*_1, g^*_2, \) and \( g^*_3 \) respectively.

Consider two networks \( g^*_k \) and \( g^*_k+l \), where \( l > 0 \), then \( g^*_k \) is a mean-preserving spread of \( g^*_k+l \) which is obtained by rewiring \( l \) links in \( g^*_k \). Note that if \( l = 1 \) then \( g^*_k \) is a one-step mean-preserving spread of \( g^*_k+l \), which is indicated by an arrow in figure 1. Finally, we will also consider the line network \( g^{--} \), which is represented at the bottom of figure 1, because any two-layer star network is a mean-preserving spread of the line network.

Denote by \( q(g) \) the long-run probability that two agents meet in network \( g \). First, let us consider the case when there are a few agents, and for simplicity let \( n = 2 \). The following remark compares the probabilities that the two agents meet in \( g^*_k \) and \( g^{--} \).

**Remark 1.** Assume that there are only two agents and \( s \) sites. Consider environments represented by a line network \( g^{--} \) and by the class of two-layer star networks \( g^*_k \). We have that \( q(g^*_k) \geq q(g^*_k+l) \geq q(g^{--}) \) for \( k \in [0, s/2) \). Specifically, for \( s \geq 3 \) we have that:

\[
q(g^*_k) = \frac{(s - k - 1)^2 + s + 3k - 1}{4(s - 1)^2} \geq \frac{1}{(s - 1)^2} \left( s - \frac{3}{2} \right) = q(g^{--})
\]

Note that \( q(g^*_k) = q(g^*_k+l) = q(g^{--}) \) holds only for \( s = 3 \), which is the case in which all networks under consideration are identical. Also, note that \( \frac{3}{2} > q(g^*) \).

The probability \( q(g^*_k) \) that the two agents meet in the two-layer star network with \( k \) sites in the outer layer is higher than the probability \( q(g^*_k+l) \) that they meet in the two-layer star network with \( k + l \) sites in the outer layer. Moreover, the probability \( q(g^*_k) \) that they meet in any two-layer star network is higher than the probability that they meet in the line network \( q(g^{--}) \).

The intuition is that if there are only two agents then most of the time an agent moves there is no meeting because the destination site is empty. Thus the presence of the central
Figure 1: The four networks at the top are all the members of the class of two-layer star networks with $s=7$ sites. Proceeding clockwise from the top-left corner we have the two-layer star networks $g^*, g_1^*, g_2^*$ and $g_3^*$. The network at the bottom is the line network, and the network on the top left is the star network. A continuous arrow indicates a one-step mean-preserving spread.
site in the two-layer star network makes it more likely that the two agents will meet at this trafficked site. On the other hand in the line network the two agents may spend a considerable amount of time in different parts of the line where any one-step movement will result in no meeting. Similarly, a two-layer star network $g^*_k$ has more sites linked to the central site than a two-layer star network $g^*_{k+l}$ so the central site is more trafficked and it is more likely that the two agents will meet there. Note that the probability of meeting is always lower than $2/n$ because it is possible that no meeting occurs after one agent has moved.

Second, let us consider the alternative case when there are many agents, and for illustrative purposes assume that $n \gg s$ in this example.

**Remark 2.** Assume that there are $n$ agents and $s \geq 3$ sites with $n \gg s$. Consider environments represented by a line network $g^{*-*}$ and by the class of two-layer star networks $g^*_k$. We have that $q(g^{*-*}) \geq q(g^*_{k+l}) \geq q(g^*_k)$ for $k \in [0, s/2)$. Note that $q(g^*_k) = q(g^*_{k+l}) = q(g^{*-*})$ holds only for $s = 3$, which is the case in which all networks under consideration are identical. Moreover, $\frac{2}{n} > q(g^{*-*})$.

The analytical expressions for $q(g^*_k)$ and $q(g^{*-*})$ are available in Appendix A. The probability $q(g^*_{k+l})$ that the two agents meet in the two-layer star network with $k + l$ sites in the outer layer is higher than the probability $q(g^*_k)$ that they meet in the two-layer star network with $k$ sites in the outer layer. Moreover, the probability that they meet in the line network $q(g^{*-*})$ is higher than the probability $q(g^*_k)$ that they meet in any two-layer star network.

The result is the opposite of what we found for the case with only two agents. If there are many agents, then most of the time the traveling agent moves to a site there are other agents and therefore a meeting occurs almost always. Thus, the key to maximizing the probability of a meeting is to minimize the probability that there is no meeting when the agent moves to one of the less trafficked sites. The crucial structural feature to minimize the probability that there is no meeting is the absence of poorly connected sites. In the line network, all sites, except for the sites at the end of the line, have two connections so the agents are spread uniformly across the line and this minimizes the probability that there is no meeting. On the other hand, in the star network all the sites, except for the central one, are poorly connected with only one link and therefore there is a non-negligible probability that an agent may end up on an unoccupied peripheral site. Thus, the probability that there is a meeting is higher in the line network than in the star network.

Similarly, if we compare two-layer star networks $g^*_{k+l}$ and $g^*_k$ with, respectively, $k + l$ and $k$ links in the outer layer then in $g^*_{k+l}$ there are more sites in the inner layer that are linked to a peripheral site as well as to the center. Thus, the probability that the sites in the inner layer are unoccupied is higher in $g^*_k$ than in $g^*_{k+l}$, and therefore the probability that there is a meeting is higher in $g^*_{k+l}$ than in $g^*_k$. Note that the probability of meeting
is always lower than $2/n$ because it is possible that no meeting occurs after one agent has moved.

Remarks 1 and 2 rank the two-layer star networks and the line network according to mean-preserving spreads. Remark 1 states that if there are only 2 agents then a mean-preserving spread of the network increases the probability of a meeting. On the other hand, remark 2 states that if there are many agents then a mean-preserving spread of the network decreases the probability of a meeting. This suggests a conjecture that this statement may extend to any network. The next section proves this conjecture.

4 Speed of learning and network structure

The first part of this section shows that in the long-term the system converges to an equilibrium where all agents hold the same belief. The second part contains the main results of the paper that show how the speed of learning depends on the underlying network structure.

4.1 Convergence

The main focus of the paper is an analysis of the speed of learning. However, this type of analysis matters only if one can show that there is some meaningful learning at the societal level. The following lemma shows that this is the case.

Lemma 1. Assume that $\theta$ is the mean of the initial signals received by the agents. There exists a random variable $x^0$ with distribution $F(x^0)$ such that:

(i) $\Pr (\lim_{t \to \infty} x^t_i = x^0) = 1$ for all $i \in N$

(ii) $\mu[F(x^0)] = \theta$

The first part of the lemma says that in the long-term everyone in the society converges (almost surely) to the same limit belief about the underlying state of the world. The second part states that on average this limit belief will be the mean of the initial signals. Thus, if the initial signals are unbiased, in the sense that their mean is equal to the true value $\theta$ of the underlying state of the world, then on average there is meaningful learning at the societal level.

The intuition is that all the agents are interchangeable so over time each one of them will have the same impact in determining the belief everyone converges to, and this leads the society to aggregate the initial signals optimally. This result is independent of the network structure because the underlying network is the same for every agent so it does not introduce any asymmetry across the agents. However, the network matters in determining the speed of convergence.
The quantity that we will compute to analyze the speed of learning is the mean time $\tau$ an agent has to wait between one meeting and the next. Formally, the quantity that we are interested in is the mean recurrence time to a subset of states of the Markov process describing this system. Consider the finite set $X$ whose elements are all the possible states $x_{ij}$ of the system. Note that the definition of a state $x_{ij}$ explicitly specifies the traveling agent $i$ and the agent $j$, if there is any, that $i$ has met. As illustrated in figure 2 below, we can consider the subset of states $X_i = \{x_{ij}, x_{ji} | j \neq 0\}$ in which agent $i$ met with another agent $j$ in the last time period. Suppose that at time $t$ the system is in state $x_{ij}$ where $i$ has just moved and met $j$ (or in state $x_{ji}$ where $j$ has just moved and met $i$), and suppose that at time $t+1$ the system is in a state $x_{pq} \notin X_i$. The mean recurrence time $\tau$ to the subset $X_i$ is then the mean time it takes the system to return to a state $x_{ik}$ (or $x_{ki}$) inside the subset $X_i$.

The following theorem states the first main result of the paper: an explicit expression that captures how the expected time an agent has to wait between two meetings depends on the underlying environment and the number of other agents. For notational convenience we will drop the expectation operator hereafter, so $\tau(n, s, g)$ denotes the expected time $E[\tau(n, s, g)]$ between two meetings.

**Theorem 1.** The expected time $\tau(n, s, g)$ agent $i$ has to wait between two meetings is equal to:

$$\tau(n, s, g) = \frac{nD}{2s \sum_{d=1}^{s-1} d \left[ 1 - \left(1 - \frac{d}{D}\right)^{n-1} \right] P(d)}$$

where $D = \sum_{k=1}^{s} d_k$.

The proof consists of three parts. First, by Kac’s Recurrence Theorem, the mean time $\tau(n, s, g)$ an agent $i$ has to wait between two meetings is equal to the inverse of the probability $q'(n, s, g)$ that the agent meets another agent. Second, the probability that an agent is at a given site depends only on the degree of the site by a known result on the theory of random walks on a graph. Third, the bulk of the proof consists in using this result to compute $q'(n, s, g)$ explicitly.
The dependence of $\tau$ on the network structure is fully captured by the degree distribution of the network. This is a consequence of a well-known result in the theory of random walks on a graph, which states that in the stationary state the probability that an agent is at a given site is proportional to the degree of the site. This result holds exactly for all networks that are undirected and unweighted. The requirement that the network is undirected ensures that a detailed balance condition holds. Using this condition it is straightforward to show that in the stationary distribution the probability that an agent is at a node depends only on the degree of that node. The requirement that system should be in the stationary state is satisfied because of the assumption that $T$ is large: the agents have been moving in the environment for a long time before they receive the signals about the underlying state of the world at time $T$, which is a natural assumption in our framework.

4.2 Comparative statics

We use the result in theorem 1 to investigate how the speed of learning depends on the network structure $g$: the shorter is the expected time between meetings, the higher is the frequency of meetings and therefore the faster is the learning process.

The result in theorem 1 is an exact result, but there is one approximation involved in the interpretation that variations in $\tau$ correspond to variations in the speed of learning. The expected time $\tau$ between meetings is the first moment of the distribution of the recurrence times, but the variance (and higher moments) of this distribution are also likely to vary depending on the network structure. In interpreting variations in $\tau$ as variations in the speed of learning we make the assumption that in the long-term the first moment is the main determinant of the speed of learning, and that the impact of variations in the higher moments are negligible. In other words, we focus on a mean-field approximation to study the learning process. In order to provide a clear separation between the exact results and their interpretation, we formally state the comparative statics results in terms of changes in $\tau$ and we discuss their implications for the speed of learning in the discussion of the results.

Recall from the formal definition in section 2 that a mean-preserving spread of the degree distribution of the network captures a specific type of change in network structure. Intuitively, a mean-preserving spread makes the environment more heterogeneous by increasing the variation in connectivity across sites. The following theorem captures

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5See most textbooks on the subject, e.g. Aldous and Fill [2002]. A concise proof is in Noh and Rieger [2004].

6An equivalent result to Kac's Recurrence Theorem for the variance (and higher moments) of the distribution of recurrence times is an unsolved problem in the theory of stochastic processes.

7Mean-field approximations are a standard tool used to investigate the evolution of stochastic systems. Examples of the application of this technique in different contexts within the economics of networks literature include Jackson and Rogers [2007a], Jackson and Rogers [2007b] and Lopez-Pintado [2008].
how the expected time $\tau$ between meetings, and therefore the speed of learning, changes with the level of heterogeneity of the environment.

**Theorem 2.** Consider two environments represented by networks $g$ and $g'$ with degree distributions $P(d)$ and $P'(d)$ respectively and such that $P'(d)$ is a mean-preserving spread of $P(d)$, then:

(i) If $n < \bar{n}$ then the expected time $\tau$ between meetings is lower in $g'$ than in $g$

(ii) If $n > \bar{n}$ then the expected time $\tau$ between meetings is lower in $g$ than in $g'$

where $\bar{n} \approx \frac{2D}{d_{\text{max}}(g')}$ and $\bar{n} \approx \frac{2D}{d_{\text{min}}(g')}$ and $d_{\text{min}}(g')$ and $d_{\text{max}}(g')$ are the minimum and maximum degree in $g'$ respectively.

If there is a low number of agents then a heterogeneous environment leads to a shorter expected time between meetings and therefore faster learning. When there are a few agents it is likely that when an agent moves to a new site there is nobody at that site and therefore no meeting occurs. In a heterogeneous environment the presence of highly connected sites increases the probability that the few agents in the market will meet because most of the paths in the network end up at one of these sites.

On the contrary, if the number of agents is high then a homogeneous environment leads to a shorter expected time between meetings and therefore faster learning. Unlike in the case with a few agents, when there are many agents the probability of a meeting is usually high. Thus, the crucial point is to minimize the probability that an agent moves to an unoccupied site. In a heterogeneous environment most of the paths in the network lead away from peripheral sites, which makes it more likely that an agent moving to one of the peripheral sites will find it unoccupied. On the other hand, in a homogenous environment the agents will be distributed more uniformly across the different sites minimizing the probability that the traveling agent moves to an unoccupied site.

The intuition behind this result is the same one as the one in remarks 1 and 2. The star is a prototypical heterogeneous environment, which becomes more homogeneous as we move to other elements of the class of two-layer star networks by increasing the number of sites $k$ connected to the peripheral sites of the star. The line is the prototypical homogeneous environment with the same number of links as the star. Remark 1 is a special case of Theorem 2(i): it shows that if the number of agents is 2 then the probability that $i$ meets $j$ is decreasing in $k$ for two-layer star networks because the probability that the two agents meet at the center of the star is increasing in the number of sites connected to the central site, and the probability that they meet in the line network is always lower than in any two-layer star networks. Remark 2 is a special case of Theorem 2(ii): it shows that the opposite is true if the number of agents is large because in the line network the agents are uniformly distributed across the different sites minimizing in this way the probability that a site is unoccupied.
Another type of structural change of the network is a change in the density of connections. Naively one may think that a more densely connected network decreases the expected time between meetings for an agent, but the proposition below shows that this is not the case.

**Proposition 1.** Consider the class of environments represented by the regular networks \( g^d \) with connectivity \( d \). The expected time \( \tau \) between meetings is invariant to changes in the connectivity of the network.

The intuition is that in a regular network all the sites have the same degree so they are all equivalent to each other. This implies that the network plays no role in differentiating across sites, so the expected time between meetings, and therefore the speed of learning, turns out to be independent of the network.

Even if the result in proposition 1 is restricted to regular networks, it tells us that the level of connectivity of the environment is not a primary determinant of the speed of learning. On the one hand a more connected environment allows agents to have access to a large number of sites from their current location. But on the other hand a more connected environment means that each agent has a large number of sites to move to, so, if two agents are at neighboring sites and one of them is picked to move, it is less likely that he will move to the site occupied by the other agent. These two effects cancel each other out in regular networks making the expected time between meetings independent from the connectivity of the environment.

The result in theorem 1 also allows us to investigate how the expected time between meetings depends on the number of agents \( n \) in the society, as the following corollary shows.

**Corollary 1.** The expected time \( \tau \) between meetings is

(i) increasing in the number of agents in the society if the number of agents \( n \) is below a threshold \( n_L \) and if \( \frac{d_s}{D} < \bar{d} \) for all sites \( s \in S \)

(ii) decreasing in the number of agents in the society if the number of agents \( n \) is above a threshold \( n_H > n_L \) and if \( \frac{d_s}{D} > \epsilon \) for all sites \( s \in S \)

The dependence on the number of agents is more nuanced because there are two effects at play. If the number of agents increases then on the one hand there are more agents to meet so the probability of moving to an empty site decreases and therefore learning is faster. On the other hand, if the number of agents increases then any agent is less likely to be picked to move or to be picked to meet the agent that has moved because the pool of agents has increased and only one meeting is possible in each time period.

The first effect dominates if the number of agents is low and if there is no site that is much better connected than all the others. When there are few agents and the environment is such that there is no prominent site on which the few agents can randomly
coordinate to meet then the probability of moving to an empty site is significant. Thus, an increase in the number of agents leads to a significant decrease in the probability to have unoccupied sites, and therefore a decrease in the expected time between meetings which leads to faster learning.

On the other hand, the first effect is negligible if there is a large number of agents and if there is no site that is very poorly connected. In a society where there are many agents and with an environment where each site has a non-negligible probability of being visited by agents, the probability of moving to an unoccupied site is negligible. Thus, the main effect of an increase in the number of agents is to decrease the probability that any given agent is picked, which leads to a decrease in the speed of learning.

However, it is worthwhile to notice that the second effect is dependant on our definition of time in the model. An alternative definition would be to normalize a ‘time step’ by the number of agents so that each agent is picked to move once in each time period. Adopting this alternative definition we would have that the expected time between meetings is always decreasing in the number of agents because only the first effect would survive.

5 Influencing the society

The role that learning by chit-chat plays in shaping the views that a society converges to means that there are individuals and/or businesses that have an interest in influencing this process. In the learning by chit-chat framework we can model influencers as individuals who never update their belief and who choose to position themselves at a specific site in order to influence the views of others, instead of moving around for reasons that are orthogonal to the learning process. This allows us to investigate which position in the network maximizes the impact of influencers, and which environments are more or less susceptible to influencers’ actions.

Specifically, consider the set-up in section 2 and assume that there is one and only one agent $i$ that we will call the influencer. Unlike the other agents who travel on the network, the influencer $i$ is positioned at one site $s_i$ and never moves away from $s_i$. If the influencer $i$ is picked to move at time $t$ then she remains on $s_i$ and she meets one randomly picked agent $k$ (if there is any) who is at site $s_i$ at time $t$. If another agent $j$ moves at site $s_i$ at time $t$ then the influencer $i$ is included in the pool of agents among which one agent is picked to meet the traveling agent $j$. Moreover, we assume that $x^t_i \equiv x_i$ for any $t$, i.e. the influencer’s belief is fixed and never changes over time.

Given that the influencer’s belief $x_i$ never changes, in the long-term the belief of all the agents in the society will converge to $x_i$. However, in most contexts the influencer does not just care about convergence per se, but how fast the convergence is. Thus, the effectiveness of the influencer in the social learning by chit-chat context is tied to the speed of convergence to $x_i$. The following definition captures this concept of effectiveness.

Definition 2. The ratio $r_i(n, s, g) = \frac{\tau(n, s, g)}{\tau_i(n, s, g)}$ is the effectiveness of the influencer $i$. An
influencer $i$ in an environment represented by $g$ is effective if $r_i(n, s, g) > 1$.

The $r_i(n, s, g)$ metric is the ratio of the mean time between meetings for a non-influencer to the mean time between meetings involving the influencer. Clearly, the higher this ratio is, the more effective the influencer is, because the frequency of meetings involving the influencer increases compared to the frequency of meetings not involving the influencer. We assume that the presence of the influencer does not affect the value of $\tau(n, s, g)$, which is a reasonable approximation given that there is only one influencer.

A natural question to investigate is to determine which position in the network maximizes the influencer’s effectiveness. The following proposition shows that the influencer achieves the maximum effectiveness by placing herself at the most connected site.

**Proposition 2.** Consider any environment represented by a network $g$. The position that maximizes the effectiveness of the influencer is at the site with the largest degree, and the effectiveness of the influencer at this site is $r_i(n, s, g) \geq 1$.

The intuition is rather straightforward. The site with the highest degree is the most trafficked one so the influencer based on this site will have the highest frequency of meetings and therefore she will be able to exert the strongest influence on the other agents. Moreover, the influencer at the most connected site will always be (weakly) effective. Hereafter we will assume that the influencer optimally chooses to be on the most connected site in order to maximize her effectiveness. The following definitions will be useful to analyze how the effectiveness of an influencer positioned at the most connected site depends on the underlying environment.

**Definition 3.** Consider two environments represented by $g$ and $g'$ and a society with one influencer $i$ positioned at the site with the largest degree. We say that environment:

- $g$ is **minimally susceptible** to the action of the influencer $i$ if $r_i(n, s, g) = 1$
- $g$ is **more susceptible** to the action of influencer $i$ than $g'$ if $r_i(n, s, g) > r_i(n, s, g')$

The first result is that there is a class of environments that are minimally susceptible to the action of the influencer.

**Proposition 3.** An environment represented by a network $g$ is minimally susceptible to the influencer if and only if $g$ is a regular network.

In a regular network all the sites are equivalent and therefore there is no site where the influencer can position herself to increase the frequency of meetings vis-à-vis the other agents. The other important point is that the influencer is effective in all environments, except for the special class of environments represented by regular networks. Thus, the learning by chit-chat process is rather susceptible to the action of an influencer who is not particularly knowledgeable or sophisticated: she behaves in the same way as all the other agents and the additional knowledge required to exert her influence is simply the knowledge of the location of the most connected site.
In general, different environments will have different levels of susceptibility to the influence of the influencer. The following proposition provides a partial ranking of environments according to their level of susceptibility to the influencer.

**Proposition 4.** Consider two environments represented by networks $g$ and $g'$ with degree distributions $P(d)$ and $P'(d)$ respectively and such that $P'(d)$ is a mean-preserving spread of $P(d)$. If $n > \bar{n} \equiv 2D/d_{\text{max}}(g')$ then $g'$ is more susceptible to the influencer than $g$.

The proof is mainly a consequence of theorem 2. Recall from section 4.2 that if the number of agents is above the threshold $\bar{n}$ then a mean preserving spread increases the mean time $\tau$ that an agent has to wait between two meetings. Moreover, by definition of a mean preserving spread, if $g'$ is a MPS of $g$ then the degree of the most connected site in $g'$ is (weakly) larger than the degree of the most connected site in $g$. The mean time $\tau_i$ the influencer has to wait between two meetings is inversely related to the degree of the most connected site, and therefore a mean preserving spread decreases the time an influencer has to wait between two meetings. Thus, the effectiveness $r_i \equiv \tau/\tau_i$ of the influencer increases with a mean preserving spread of the underlying network.

Intuitively, there are two effects that explain the increase in the influencer’s effectiveness. The first one is the same effect that drives the result in theorem 2(ii). If there are many agents then the key to shorten the mean time between meetings is to minimize the probability that an agent moves to an unoccupied site. A shift to a more heterogeneous environment increases the number of poorly connected sites and therefore it increases the probability of moving to an unoccupied site. Note that this does not affect the influencer, who is always at the most connected site whose probability of being unoccupied is negligible. If the probability of moving to an unoccupied site increases then the frequency of meetings that do not involve the influencer decreases, which helps to increase the effectiveness of the influencer. The second effect is that a shift to a more heterogeneous environment (weakly) increases the connectivity of the most connected site where the influencer is. This means that the traffic on the site where the influencer is increases and therefore the frequency of meetings involving the influencer increases as well.

If the number of agents is low then it is not feasible to rank the environments according to their susceptibility to the influencer because the first effect above is reversed, and therefore the two effects go into opposite directions. When there are a few agents, the result of theorem 2 says that a shift to a more heterogenous environment leads to an increase in the frequency of meetings for any traveling agent because of an increase in the probability of meeting on a trafficked site. But the direction of the second effect is unchanged: a more heterogeneous environment increases the connectivity of the most connected site leading to an increase in the frequency of the influencer who is located at that site. Thus, here the first effect decreases the effectiveness of the influencer while the second one increases it, and therefore the overall effect of a mean preserving spread of the network will depend on the details of the two distributions.
When the number of agents is low there are examples of MPS changes in the network in which the effectiveness of the influencer can go up, and examples in which it can go down. For an example of a MPS shift in which the effect of the influencer goes down, consider two networks $g$ and $g'$ such that $g'$ MPS g and $d_{\text{max}}(g) = d_{\text{max}}(g')$. It is straightforward to show that in this case $\tau_i(n,s,g) = \tau_i(n,s,g')$, and by the result of theorem 2 we have that $\tau(n,s,g) > \tau(n,s,g')$. Thus, substituting these inequalities into the definition of effectiveness we have that $r_i(n,s,g) > r_i(n,s,g')$.

6 Conclusion

This paper formulated a novel framework to study a type of learning process that we dub social learning by chit-chat. The main features of this learning process are that individuals learn as a by-product of their daily activities, they do not rely exclusively on their social relations to learn, and the speed of learning depends on the number of individuals in the population and on the environment in which they are embedded. Individuals are located on different nodes of a network, which represents an underlying environment. At time 0, they receive a noisy signal about the underlying state of world. In each successive period an individual is randomly picked to move to a new location, if there is at least another individual at the destination then a meeting occurs and individuals update their beliefs after having learnt the belief of the individual they have met.

In the long-term everyone in the society holds the same belief. The distribution of this limit belief is symmetric around a mean which is equal to the average of the initial signals. We derive an exact formula that describes how the expected time between meetings depends on the number of individuals and the underlying environment. We interpret results on the variations in $\tau$ as variations in the speed of learning by using a mean-field approximation. If the number of individuals is below (above) a threshold then a mean preserving spread of the network increases (decreases) the speed of learning. Moreover, the speed of learning is the same in all regular networks, irrespective of their degree. An extension analyzes whether an agent, dubbed the influencer, who does not change her belief and stations at one location, is able to influence the learning process and how the effectiveness of the influencer depends on the underlying network structure.
A Appendix: Proofs

This appendix contains the proofs of all the statements made in the main body.

**Proof of Remark 1.** There are two ways for agent $i$ to have a meeting with $j$: either $i$ is picked to move and travels to the site where $j$ is, or $j$ is picked to move and she moves to the site where $i$ is. By switching the labels $i$ and $j$ it is particularly easy to see here that the probabilities of these two events are the same, hence the probability $q(g)$ that $i$ has a meeting is equal to twice the probability $q_1(g)$ that $i$ is picked to move and travels to the site where $j$ is.

For the line network we have that the probability $q_1(g_{--})$ that $i$ is picked to move and travels to the site where $j$ resides is:

$$q_1(g_{--}) = \frac{1}{2} \left\{ 2 \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2(s-1)} \right] + 2 \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{2}{2(s-1)} \right] + \frac{1}{2} \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2(s-1)} \right] \right\}$$

$$+ \frac{1}{2} \left\{ (s-4) \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{2}{2(s-1)} + \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{2}{2(s-1)} \right] \right\}$$

where the $1/2$ factor upfront is the probability that $i$ is selected; the three terms in square brackets are the probabilities that $i$ meets $j$ after moving to the node at the end of the line, to the neighbor of the node at the end of the line and to one of the remaining nodes in the middle respectively. Each term in the square bracket is composed by the probability that $i$ is at the starting node, times the probability that $i$ moves to the node where $j$ is, times the probability that $j$ is at the destination node. The number of terms in the square bracket indicates the number of links to reach the destination site where $j$ is, and the factor in front of the square brackets is the number of the type of destination nodes (e.g. there are two nodes at each end of the line). Summing up we obtain that:

$$q(g_{--}) = 2q_1(g_{--}) = \frac{1}{(s-1)^2} \left( \frac{s-3}{2} \right)$$

Similarly, for the two-layer star network $g_k^*$ we have that:

$$q_1(g_k^*) = \frac{1}{2} \left\{ (s-2k-1) \left[ \frac{1}{2(s-1)} \cdot \frac{s-k-1}{2(s-1)} \right] + k \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{s-k-1}{2(s-1)} \right] \right\} +$$

$$+ \frac{1}{2} \left\{ (s-2k-1) \left[ \frac{s-k-1}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2(s-1)} \right] \right\} +$$

$$+ \frac{1}{2} \left\{ k \left[ \frac{1}{2(s-1)} \cdot \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{s-k-1}{2(s-1)} \right] \right\} +$$

$$+ \frac{1}{2} \left\{ k \left[ \frac{2}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2(s-1)} \cdot \frac{2}{2(s-1)} \right] \right\}$$

21
where the expressions in each line are, respectively, the probabilities that \( i \) meets \( j \) after moving to the center node, to a node in the inner layer that is not linked to an outer layer node, to a node in the inner layer that is linked to an outer layer node, and to a node in the outer layer. The explanation for the various terms in the brackets is the same as above. Summing up we obtain that:

\[
q(g^*_k) = 2q_1(g^*_k) = \frac{(s-k-1)^2 + s + 3k - 1}{4(s-1)^2}
\] (4)

From equations (3) and (4) it is easy to see that \( q(g^*_k) > q(g^*_{k+l}) > q(g^{**}) \). Note that if \( s = 3 \) then the star \( g^* \) is the only two-layer network and \( q(g^*) = q(g^{**}) \), as expected because if \( s = 3 \) then the star and the line network are the same network.

**Proof of Remark 2.** For the line network we have that the probability \( q_1(g^{**}) \) that \( i \) has a meeting after he was picked to move is:

\[
q_1(g^{**}) = \frac{2}{n} \left\{ \frac{1}{2} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \left[ 1 - \left( 1 - \frac{1}{2(s-1)} \right)^{n-1} \right] \right\} + \\
\frac{s-4}{n} \left\{ \frac{2}{2} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2} \cdot \left[ 1 - \left( 1 - \frac{2}{2(s-1)} \right)^{n-1} \right] \right\} + \\
\frac{2}{n} \left\{ \frac{1}{2} \cdot \frac{1}{2(s-1)} \cdot \frac{1}{2} \cdot \left[ 1 - \left( 1 - \frac{2}{2(s-1)} \right)^{n-1} \right] \right\}
\]

where the three terms are the probability that \( i \) has a meeting after moving to a node at the end of the line, in the middle of the line and a node neighboring a node at the end of the line. The factor in front of the curly brackets is the probability that \( i \) is selected times the number of nodes of that type in the line network. The term in square bracket is the probability that there is at least another agent at \( i \)'s destination node, multiplied by the following factors: the number of nodes from which \( i \) could move to the destination node, the probability that \( i \) is at one of the departure nodes, and the probability that \( i \) moves to the destination node from the departure node. Summing and simplifying we obtain:

\[
q_1(g^{**}) = \frac{1}{n(s-1)} \left\{ \left[ 1 - \left( 1 - \frac{1}{2(s-1)} \right)^{n-1} \right] + (s-2) \left[ 1 - \left( 1 - \frac{1}{s-1} \right)^{n-1} \right] \right\}
\] (5)

Following a similar procedure for the two-layer star network with \( k \in [0, s/2] \) sites in the outer layer we obtain that the probability \( q_1(g^*_k) \) that \( i \) has a meeting after he was
picked to move is equal to:

\[ q_1(g_k^s) = \frac{s - k - 1}{2n(s - 1)} \left[ 2 - \left( 1 - \frac{s - k - 1}{2(s - 1)} \right)^{n-1} - \left( 1 - \frac{1}{2(s - 1)} \right)^{n-1} \right] + \frac{k}{n(s - 1)} \left[ 1 - \left( 1 - \frac{1}{s - 1} \right)^{n-1} \right] \]

(6)

Note that for \( s = 3 \) we have that \( q_1(g^{**}) = q_1(g_k^s) \), which is what we would expect because when \( s = 3 \) the star network is the unique two-layer star network and it is the same network architecture as the line network.

By a symmetry argument, or by explicit computation as carried out in the proof of theorem 1 below, we have that \( q(g_k^s) = 2q_1(g_k^s) \) and \( q(g^{**}) = 2q_1(g^{**}) \). Differentiating (6) with respect to \( k \) we can see that \( q(g_k^s) \) is decreasing in \( k \) and therefore we have that \( q(g_k^s) > q(g_{k+1}^s) \). Finally, a comparison of expressions (5) and (6) shows that if \( n \gg s \) then \( q(g_k^s) > q(g^{**}) \) for all \( k \in [0, s/2] \).

\[ \square \]

**Proof of Lemma 1.** Define \( x_L^t, x_H^t : \Omega \rightarrow \mathbb{R} \) to be two random variables that capture the beliefs of the agents with the lowest and highest beliefs at time \( t \), and where \( \Omega \) is the space of all sample paths of the system. Let \( \Delta t = x_H^t - x_L^t \). Let us prove each statement separately.

(i) First, we prove that \( x_H^t \) is decreasing over time. Suppose that at time \( t \) agent \( i \) has the highest belief \( x_H^t = x_i^t \) for all \( j \neq i \) and that agent \( i \)'s next meeting is at time \( t' > t \). By definition at time \( t \) we have that \( x_H^t = x_i^t \). At any time step \( t'' \) such that \( t < t'' < t' \) we claim that \( x_H^t = x_i^t \). This follows because no agent \( j \neq i \) can have beliefs \( x_j^t < x_i^t \).

There are two possible cases. If \( x_j^t = x_i^t \) then \( x_j^t = x_j^t < x_i^t = x_i^t \). If \( x_j^t \neq x_i^t \) then it must be that \( j \) met one or more agents \( k \neq i \), assume without loss of generality that \( j \) met one agent \( k \) then \( x_j^t = x_j^t - \alpha(x_k^t - x_j^t) < x_i^t = x_i^t \) because \( x_k^t < x_i^t \) and \( x_j^t < x_i^t \). Given that at time \( t'' \) no agent \( j \) has beliefs \( x_j^t < x_i^t \) and that \( i \) has no meeting between \( t \) and \( t'' \) then it must be that \( x_H^t = x_i^t \). Now consider time \( t' \) when agent \( i \) meets agent \( j \) who has beliefs \( x_j^t < x_i^t \) then we have that \( i \)'s beliefs at time \( t' \) are equal to \( x_H^t = x_i^t - \alpha(x_j^t - x_i^t) < x_i^t \). Thus, \( x_H^t \) is decreasing over time. A similar argument shows that \( x_L^t \) is increasing over time.

Second, we show that \( Pr(\lim_{t \to \infty} x_i^t = x^0) = 1 \) for all \( i \in N \). Pick any \( \epsilon > 0 \) and suppose that at time \( T' \) we have that \( \Delta T' = K \epsilon \) for \( K > 1 \). Define \( p \equiv \frac{1}{2 \alpha} \left( 1 - \frac{1}{R} \right) \). With probability 1, there is a time \( T'' > T' \) such that the agents with the highest and lowest belief have met at least \( p \) times since time \( T' \). It is then straightforward to see that at
time $T''$ we have that:

$$\Delta^{T''} \leq k\epsilon[1 - 2p\alpha + O(\sum_{q=1}^{n} \alpha^q)] \approx k\epsilon[1 - 2p\alpha] = \epsilon$$

where $q \in \mathbb{N}$. This proves the first part of the statement.

(ii) Fix the allocation of agents to sites at time $t = T$. Without loss of generality, assume that the average of the $n$ initial signals is equal to $\theta$ and allocate one signal to each agent. By part (i) above, as $t \to \infty$ we have that every agent’s belief converges (almost surely) to $x^0$. Suppose that $x^0 = \theta + \delta$ for any $\delta \neq 0$. If this is the case it must be that a subset of agents were more influential in the learning process, because if all the agents had the same influence then the limit belief would be the average $\theta$ of the initial signals.

Now consider a different allocation of initial signals such that if agent $j$ had received signal $\theta_j$ in the allocation considered above, then in the new allocation it receives a signal $-\theta_j$. By the symmetry of the distribution of initial signals, the two allocations have the same probability of occurrence. Given that the underlying network $g$ is the same, there is also the same probability that the same meetings occur in the same order and therefore as $t \to \infty$ we have that every agent has the same belief $x^0 = \theta - \delta$. Thus, the distribution $F(x^0)$ of the limit belief $x^0$ is symmetric around a mean $\mu[F(x^0)] = \theta$.

\[\square\]

**Proof of Theorem 1.** Let $X$ be the set of states of the system, and let $x_{ij}^t(s)$ be the state in which $i$ moves to site $s$ where $i$ meets $j$ at time $t$. It is clear that the stochastic process describing the evolution of the system is strongly stationary. Consider a completely additive probability measure $q$ on $X$, i.e. $q(X) = 1$

Let $A$ be a subset of states of the stochastic process, i.e. $x_{ij}^t(s) \in A \subset X$. It follows from strong stationarity that $P\{x_{ij}^t(s) \in A\} = q\{x_{ij}^t(s) \in A\}$ is the same for all $t$. Given that it is independent of $t$ and for notational convenience, let $P\{x_{ij}^t(s) \in A\} \equiv P(A)$. Similarly, let $P\{x_{ij}^t(s) \notin A\} \equiv P(\overline{A})$. Furthermore, define the following quantities:

$$P^{(n)}(A) = P\{x_{ij}^t(s) \in A, x_{ij}^{t+1}(s) \notin A, ..., x_{ij}^{t+n-1}(s) \notin A, x_{ij}^{t+n}(s) \in A\}$$

$$P^{(n)}(\overline{A}) = P\{x_{ij}^t(s) \notin A, x_{ij}^{t+1}(s) \notin A, ..., x_{ij}^{t+n}(s) \notin A\}$$

and

$$Q^{(n)}(A) \equiv P^{(n)}[x_{ij}^{t+n}(s) \in A | x_{ij}^t(s) \in A, x_{ij}^{t+1}(s) \notin A] = \frac{P^{(n)}(A)}{P(A)}$$

Finally, define the mean recurrence time $\tau(A)$ to the subset $A$ to be equal to:

$$\tau(A) = \sum_{n=2}^{\infty} Q^{(n)}(A) = \sum_{n=2}^{\infty} P^{(n)}(x_{ij}^{t+n}(s) \in A | x_{ij}^t(s) \in A, x_{ij}^{t+1}(s) \notin A)$$
Kac’s Recurrence Theorem (KRT) proves that if \( \lim_{n \to \infty} P_n(A) = 0 \) then \( (i) \sum_{n=2}^{\infty} Q_n(A) = 1, \) and \( (ii) \) the mean recurrence time to the subset \( A \) is equal to:

\[
\tau(A) = \frac{1}{P(A)} = \frac{1}{q(A)}
\]

See Kac [1957] for a statement and a proof of this result. Informally, this statement says that if the probability of the system being in the subset \( A \) is nonzero, then the expected time for the system to return to a state in \( A \) is equal to the inverse of the probability of being in the subset of states \( A \). Thus, in order to compute \( \tau(n, s, g) \) we need to compute the probability that the system is in a state in which a given agent \( j \) has a meeting.

In computing this probability we make use of a result from the theory of random walks on a graph: an independent random walker on an unweighted, undirected network spends a proportion of time \( d_c/D \) at node \( c \), i.e. a proportion of time proportional to the degree of the node. The fact that in the stationary state the probability that a random walker is at a node depends on a simple metric like the degree of the node is a result that holds exactly for unweighted and undirected networks.\(^9\) Note that the system is in stationary state because \( T \) is large.

There are two ways in which \( j \) can meet another agent. The first one is that there is a probability \( q_{jc}^1 \) that he moves to a site \( c \) where there is at least one other agent. The second one is that there is a probability \( q_{jc}^2 \) that another agent moves to site \( c \) where \( j \) currently is and that \( j \) is selected to trade with the newcomer. Let us compute these two probabilities.

The probability that \( j \) is selected to move to a new site \( c \) is \( 1/n \) and, using the result from the theory of random walkers on a graph, the probability that \( c \) is the site that \( j \) ends up at is \( d_c/D \). These two terms multiplied by the probability that there is at least one other agent at \( c \) give the following expression for \( q_{jc}^1 \):

\[
q_{jc}^1 = \frac{1}{n} \frac{d_c}{D} \left[ 1 - \left( 1 - \frac{d_c}{D} \right)^{n-1} \right]
\]

The expression for \( q_{jc}^2 \) is similar to the one for \( q_{jc}^1 \) above. The second and third term of (8) are unchanged because they capture the probability that the selected agent \( k \) ends up at site \( c \) where there is at least one other agent. However, now we have to multiply these two terms by the probability that the site where \( j \) currently resides is the one where \( k \) moved to and that \( j \) is picked to trade with \( k \). Consider the system after \( k \) has moved and let \( m \) be the number of agents on the site \( c \) where \( j \) is. Given that \( j \) is on \( c \), the

\(^9\)The requirements that the network is undirected and unweighted ensure that a detailed balance condition holds. Specifically, \( d_a p_{ab}(t) = d_b p_{ba}(t) \) where \( p_{ab}(t) \) is the probability that an agent moves from \( a \) to \( b \) in \( t \) time steps. See Noh and Rieger [2004] for a concise proof, or most standard textbooks on the subject (e.g. Aldous and Fill [2002]).
probability that $k$ is among those agents is $m - 1/n$ and the probability that $j$ is picked to trade with $k$ is $1/m - 1$. This gives the following expression for $q^{j_c}_2$:

$$q^{j_c}_2 = \frac{1}{m - 1} \frac{n - 1}{n} \left[ 1 - \left( 1 - \frac{d_c}{D} \right)^{n-1} \right] = \frac{1}{n D} \left[ 1 - \left( 1 - \frac{d_c}{D} \right)^{n-1} \right]$$  \hspace{1cm} (9)

Summing up (8) and (9) we obtain the probability $q^j$ that $j$ is involved into a meeting at site $c$:

$$q^j = \frac{2}{m - 1} \frac{d_c}{n D} \left[ 1 - \left( 1 - \frac{d_c}{D} \right)^{n-1} \right]$$  \hspace{1cm} (10)

The probability $q^j(d_c)$ that $j$ is involved into a meeting at a site of degree $d_c$ is then equal to:

$$q^j(d_c) = \frac{2}{n D} \left[ 1 - \left( 1 - \frac{d_c}{D} \right)^{n-1} \right] s \cdot P(d_c)$$  \hspace{1cm} (11)

Taking the expectation over the degree distribution of the network $g$ we obtain the probability $q^j$ that $j$ is involved into a meeting:

$$q^j = \frac{2 s}{n D} \sum_{d=1}^{s-1} d \left[ 1 - \left( 1 - \frac{d}{D} \right)^{n-1} \right] P(d)$$  \hspace{1cm} (12)

Finally, applying Kac’s Recurrence Theorem we obtain the mean waiting time $\tau(n, s, g)$ that a given agent $j$ has to wait before the next meeting:

$$\tau_j(d) = \frac{1}{q^j} = \frac{n D}{2 s \sum_{d=1}^{s-1} d \left[ 1 - \left( 1 - \frac{d}{D} \right)^{n-1} \right] P(d)}$$  \hspace{1cm} (13)

**Proof of Theorem 2.** Recall that from (12) the probability $q^j$ that the system is in a state where $j$ interacts with another agent is equal to:

$$q^j = \frac{2 s}{n D} \sum_{d=1}^{s-1} d \left[ 1 - \left( 1 - \frac{d}{D} \right)^{n-1} \right] P(d) = \frac{2 s}{n D} \sum_{d=1}^{s-1} f(d) P(d)$$  \hspace{1cm} (14)

Differentiating $f(d)$ we have:

$$f'(d) = \frac{\partial f(d)}{\partial d} = 1 + \frac{D}{(d - D)^2} (dn - D) \left( 1 - \frac{d}{D} \right)^n$$  \hspace{1cm} (15)

$$f''(d) = \frac{\partial^2 f(d)}{\partial d^2} = \frac{D(n - 1)}{(d - D)^3} (dn - 2D) \left( 1 - \frac{d}{D} \right)^n$$  \hspace{1cm} (16)
Now consider each case separately.

(i) Assume that $n < \bar{n} = \frac{2D}{d_{\text{max}}} - 1$. By inspection of (15) and (16) it is straightforward to see that $f'(d) < 0$ and $f''(d) > 0$, i.e. $f(d)$ is strictly decreasing and convex in $d$. This and the fact that $P'(d)$ is a strict mean-preserving spread of $P(d)$ imply that:

$$q^j = \sum_{d=1}^{s-1} f(d) P(d) < \sum_{d=1}^{s-1} f(d) P'(d) = q'^j$$

and the result follows after application of Kac’s Theorem.

(ii) Assume that $n > \bar{n} = \frac{2D}{d_{\text{min}}}$. By inspection of (15) and (16) it is straightforward to see that $f'(d) > 0$ and $f''(d) < 0$, i.e. $f(d)$ is strictly increasing and concave in $d$. This and the fact that $P'(d)$ is a strict mean-preserving spread of $P(d)$ imply that:

$$q^j = \sum_{d=1}^{s-1} f(d) P(d) > \sum_{d=1}^{s-1} f(d) P'(d) = q'^j$$

and the result follows after application of Kac’s Theorem.

Proof of Proposition 1. In the special case of a regular network $g^d$ with degree $d$ we have that $D = d \cdot s$ and $P(d)$ is a point mass at $d$, so by simple substitution in equation (2) we obtain that:

$$\tau(n, s, g^d) = \frac{n}{2s \left[ 1 - \left(1 - \frac{1}{s} \right)^{n-1} \right]}$$

Clearly (19) is independent of $d$ so in regular networks the mean time $\tau(n, s, g^d)$, and therefore the speed of learning, is invariant to changes in the connectivity $d$ of the network.

Proof of Corollary 1. Differentiating (12) with respect to $n$ we have that:

$$\frac{\partial q^j(n, s, g)}{\partial n} = \frac{2s}{nD^2} \sum_{d=1}^{s-1} d \left\{ \left[ \left(1 - \frac{d}{D} \right)^n - 1 \right] - n \left(1 - \frac{d}{D} \right)^{n-1} \log \left(1 - \frac{d}{D} \right) \right\} P(d)$$

Note that the first term is always negative and the second term is always positive. Let $n < n_L$ and $\frac{d}{D} < \bar{d}$ then it is straightforward to check, by numerical evaluation or algebraically, that the second term in the numerator is larger than the first term, and therefore we have that $\frac{\partial q^j(n, s, g)}{\partial n} > 0$. Thus the probability of a meeting is increasing with the number of agents and therefore, after application of KRT, the expected time between meetings decreases with the number of agents $n$.

On the other hand if $n > n_H > n_L$ and $\frac{d}{D} > \epsilon$ then the first term in the numerator is larger than the second term, so we have that $\frac{\partial q^j(n, s, g)}{\partial n} < 0$. Thus the probability of a meeting is decreasing with the number of agents and therefore, after application of KRT, the expected time between meetings increases with the number of agents $n$. 

27
Proof of Proposition 2. Suppose that the influencer $i$ is at site $z$ in the network. Similarly to a non-influencer, there are two ways in which $i$ can meet another agent. The first one is that there is a probability $q_{iz}^{1}$ that $i$ is selected and she meets another agent if there is at least one other agent on site $z$. The second one is that there is a probability $q_{iz}^{2}$ that another agent $j$ moves to site $z$ and $i$ is the agent picked to meet the newcomer.

The probability that $i$ is selected to move to a new site is $1/n$. By definition the influencer stays at site $z$ so the probability of a meeting $q_{iz}^{1}$ is given by the probability that the influencer is selected times the probability that there is at least one other agent at $z$:

$$q_{iz}^{1} = \frac{1}{n} \left[ 1 - \left(1 - \frac{d_z}{D}\right)^{n-1} \right]$$

(21)

The expression for the probability $q_{iz}^{2}$ that an agent $j$ moves to site $z$ and the influencer $i$ meets the newcomer is also given by (21). It is possible to see this by a symmetry argument, or by the explicit argument contained in the proof of theorem 1. Thus, the probability $q_{iz}$ that the influencer $i$ has a meeting if she is located on site $z$ is equal to:

$$q_{iz} = q_{iz}^{1} + q_{iz}^{2} = \frac{2}{n} \left[ 1 - \left(1 - \frac{d_z}{D}\right)^{n-1} \right]$$

(22)

Applying Kac’s Recurrence Theorem we obtain that the mean time $\tau_{i}(n,s,g;z)$ the influencer positioned at site $z$ has to wait between two meetings is equal to:

$$\tau_{i}(n,s,g;z) = \frac{1}{q_{iz}} = \frac{n}{2 \left[ 1 - \left(1 - \frac{d_z}{D}\right)^{n-1} \right]}$$

(23)

Using (2) and (23) we obtain that the effectiveness $r_{i}(n,s,g;z)$ of influencer $i$ at site $z$ is equal to:

$$r_{i}(n,s,g;z) = \frac{\tau(n,s,g)}{\tau_{i}(n,s,g;z)} = \frac{D \left[ 1 - \left(1 - \frac{d}{D}\right)^{n-1} \right]}{s \sum_{d=1}^{s-1} d \left[ 1 - \left(1 - \frac{d}{D}\right)^{n-1} \right] P(d)}$$

(24)

Let $w \in S$ be the site with the maximum degree $d_{\text{max}}(g) = d_w$. It is clear by simple inspection of (24) that $\max_{z \in S} r_{i}(n,s,g;z) = r_{i}(n,s,g;w)$ and therefore the position that maximizes the influencer’s effectiveness is the site $w$ with the largest degree.

Proof of Proposition 3. First, assume that $g$ is a regular network $g^d$ of degree $d$. By the definition of a regular network it follows that (i) $D = d \cdot s$, (ii) $P(d)$ is a point mass at $d$, and (iii) $\max_{z \in S}[d_z(g)] = d$. Replacing (i)-(iii) in (24) we have that:

$$r_{i}(n,s,g^d;z) = \frac{\tau(n,s,g)}{\tau_{i}(n,s,g;z)} = 1$$

(25)
and therefore any regular network $g^d$ is minimally susceptible to the influencer.

In order to prove the other direction, let us proceed by contradiction. Assume that $g$ does not belong to the class of regular networks $g^d$ and that $r_i(n, s, g; w) = 1$ where $w$ is the site with the maximum degree. By (24) it must be that:

$$D \left[ 1 - \left( 1 - \frac{d_w}{D} \right)^{n-1} \right] = \sum_{d=1}^{s-1} d \left[ 1 - \left( 1 - \frac{d}{D} \right)^{n-1} \right] P(d)$$

and note that by definition of $w$ we have that $d_w \geq d_s$ for all $s \in S$. Thus, the only way for the equality to be satisfied is that $d_s = d_w$ for all $s \in S$, which contradicts the fact that $g$ does not belong to the class of regular networks.

**Proof of Proposition 4.** Denote by $d_{\text{max}}(g)$ the degree of the site with the largest degree in $g$. By the definition of MPS we have that if $g'$ MPS $g$ then $d_{\text{max}}(g') \geq d_{\text{max}}(g)$ and $D(g) = D(g')$. Thus, from (23) we have that:

$$\tau_i(n, s, g) = \frac{n}{2} \left[ 1 - \left( 1 - \frac{d_{\text{max}}(g)}{D(g)} \right)^{n-1} \right] \geq \frac{n}{2} \left[ 1 - \left( 1 - \frac{d_{\text{max}}(g')}{{D(g')}} \right)^{n-1} \right] = \tau_i(n, s, g') \quad (26)$$

By the result in theorem 2 we have that $\tau(n, s, g) < \tau(n, s, g')$. Thus, substituting this inequality and (26) into the definition of $r_i(n, s, g)$ we have that:

$$r_i(n, s, g) = \frac{\tau(n, s, g)}{\tau_i(n, s, g)} < \frac{\tau(n, s, g')}{\tau_i(n, s, g')} = r_i(n, s, g') \quad (27)$$

which is the desired result. \qed
References


