The Relationship between VAR and DSGE Models when Agents have Imperfect Information*

PAUL LEVINE
UNIVERSITY OF SURREY
p.levine@surrey.ac.uk

JOSEPH PEARLMAN
CITY UNIVERSITY LONDON
joseph.pearlman.1@city.ac.uk

BO YANG
SWANSEA UNIVERSITY
bo.yang@swansea.ac.uk

August 5, 2018

Abstract

How informative is a data VAR as a guide for (or even a test of) the impulse response functions in DSGE models? If we start by assuming that the DGP is our chosen DSGE model there are then two issues: the first is *fundamentalness (invertibility)*. Let $u_t = Q\epsilon_t$ be the reduced form errors in the data VAR. Are ϵ_t actually the fundamental shocks in the DSGE model? If the answer is yes we can proceed to the second issue of *identification*, the choice of the identification matrix Q using impact or long-run or sign restrictions etc. Although many concerns regarding VARs are about identification, the focus of this paper focus is on fundamentalness. We highlight the choice of information assumptions of agents in a DSGE model as an important source of non-fundamentalness. We show that validating a DSGE model by comparing its impulse response functions with those of a data VAR is more problematic when we drop the assumption that agents have perfect information. We develop measures of approximate fundamentalness for both perfect and imperfect information cases and illustrate our results using a simple analytical example and numerical examples for an RBC model and the Smets-Wouters (2007) NK model. We also show how our results impact on Bayesian estimation.

JEL Classification: C11, C18, C32, E32.

Keywords: Invertibility, VARMA, VAR, perfect versus imperfect information

*This paper was presented at the seventh annual conference of the Centre for Economic Growth and Policy (CEGAP), Durham University held on 19-20 May, 2018. We acknowledge useful comments by participants at that Conference. We have also benefited hugely from detailed written comments by Ron Smith on an earlier draft of the slides and on the paper itself.

Contents

1 Introduction

How informative is a data VAR as a guide for (or even a test of) the impulse response functions in DSGE models? If we start by assuming that the DGP is our chosen DSGE model there are then two issues: the first is *fundamentalness (invertibility)*. Let $u_t = Q\epsilon_t$ be the reduced form errors in the data VAR. Are ϵ_t actually the fundamental shocks in the DSGE model? If the answer is yes we can proceed to the second issue of *identification*, the choice of the identification matrix Q using impact or long-run or sign restrictions etc. Although many concerns regarding VARs are about identification, the focus of this paper focus is on fundamentalness.[1]

We examine the VARMA arising from the solution of a general DSGE rational expectations model and we distinguish between the information sets available to agents and to econometricians. The conventional approach implicit in the standard Bayesian estimation procedure is that all agents have perfect information about variables and shocks, whereas the econometrician has only a limited information set we refer to as imperfect information. Here by contrast our focus is on the case when agents and econometricians have the same imperfect information set, and we ask the following question that compares invertibility for the two scenarios. Suppose agents have perfect information and the information set available to the econometrician is such that the process is invertible; if by contrast the agents have the same information set as the econometrician, is the process still invertible?

Fernandez-Villaverde *et al.* (2007) and Baxter *et al.* (2011) have recently evaluated the invertibility of rational expectations models. The former have done this within the context of a reduced form, so that the main analysis involves a backwards-looking state space approach, equivalent to a set of vector VARMA processes. The reduced form can encompass perfect or imperfect information for agents, but the authors do not ask a question of the type we have posed; they are focused on general conditions for invertibility of the vector VARMA reduced form. The only information set mentioned in Fernandez-Villaverde *et al.* (2007) is that which is available to the econometrician, so that their general results are applicable no matter what is the information set of private agents. Fernandez-

[1] An excellent recent survey of the relationship between VAR and DSGE models is provided by Giacomini (2013). In common with the literature, this survey examines the issue without examining the information assumptions of the agents in the underlying DGP, the DSGE model. This is the main contribution of our paper.

Villaverde *et al.* (2007) have two main results: firstly a necessary rank condition, which states that the immediate impact of shocks on the observed variables must be of full rank; secondly they point out that if the MA part of the model does not have all eigenvalues of its characteristic polynomial outside the unit circle, then recovering the shocks directly from the observables is impossible because of instability issues. However, utilization of the Kalman filter leads to the possibility of recovering the innovations process because its steady state representation guarantees stability.

Baxter *et al.* (2011) briefly discuss the difference between instantaneous and asymptotic invertibility[2], and make no distinction between information sets of agents and econometricians (exactly as in Pearlman *et al.* (1986)). They show how non-invertibility can affect impulse response functions, and this latter is closely related to the work of Collard and Dellas (2004) and Collard and Dellas (2006). Their main result on invertibility has an almost identical form to that of Fernandez-Villaverde *et al.* (2007), but makes the crucial assumption that agents have full current information on all variables for which forward expectations are present in the model. This latter assumption is not necessarily the case; consider the standard New Keynesian model with Calvo contracts. Firms make a decision on the optimal price, but they might in principle not know the overall price index. Thus inflation might not be directly observable, even though in the model it is a forward-looking variable. Furthermore, as Sims (2002) has pointed out, the actual forward-looking variables in the Blanchard and Kahn (1980) setup are in general linear combinations of the variables with forward expectations.

The main characteristic of these papers is that non-invertibility (also referred to as "non-fundamentalness" in the literature) occurs in the following circumstances: (1) rather obviously, when the number of observables is less than the number of shocks; (2) also fairly obviously, when some observable variables of the system are observed with a lag and (3) in many DSGE models which feature anticipated shocks with a delayed effect on the system such as "news" shocks.

Here in this paper we focus mainly on models without features (1)- (3) and pose the following question: if a subset of observables is made available to the econometrician when there is perfect information for agents, and if the VARMA process corresponding to this

[2]In the time series literature, the latter is usually referred to just as invertibility.

is invertible, then the same will hold true when this is the *only* set of observables available to agents (i.e. when the situation is one of agents' imperfect information)? What we show in this paper is that this is not necessarily the case.

A recent literature on non-invertibility has emerged including recent papers by Beaudry *et al.* (2016) and Forni *et al.* (2016). They show how a particular shock of interest can be recovered, either exactly or with a good approximation, by means of standard VAR techniques even when the invertibility conditions fail and provide a diagnostic that quantifies the quality of such an approximation. It would be of interest to investigate how different information assumptions on the part of agents in a DSGE model being validated by a VAR affect this measure, but this lies beyond the scope of our paper.

The rest of the paper is organized as follows. Section 2 provides a brief survey of the related literature. Section 3 summarizes the results of Fernandez-Villaverde *et al.* (2007). Section 4 discusses the setup of a general model under rational expectations with imperfect information, and summarizes the results of Pearlman *et al.* (1986). We show that RE systems can always be transformed into Blanchard and Kahn (1980) form under perfect information, and show how this result is modified under imperfect information. The main result of this section is then a condition that is required for the system to be invertible when the number of shocks is equal to the number of observables; when invertibility holds, then of course the imperfect information solution is equivalent to full information. Section 5 relates our work to the cases when there are idiosyncratic shocks. Section 6 illustrates the key result of Section 3 using a real business cycle (RBC) model. Section 7 concludes.

2 Background Literature

Apart from the papers on invertibility/fundalmentalness discussed in the introduction, three other literatures are related to our paper. The first is *imperfect information (II) in a representative agent model*. This was initiated by Minford and Peel (1983) and generalised by Pearlman *et al.* (1986), with major contributions by Collard and Dellas (2004) and Collard and Dellas (2006), who showed that II can act act as an endogenous persistence mechanism in the business cycle. Applications of this to estimation were made by Collard *et al.* (2009), Neri and Ropele (2012) and Levine *et al.* (2012a).

The second is *II with heterogenous agents.* This distinguishes local (idiosyncratic) information and (aggregate) information. Nimark (2008), Nimark (2014), Ilut and Saijo (2016) and Graham and Wright (2010). Nimark's pioneering work shows how to generate solutions when agents have diverse information sets, and allows for the possibility of hierarchies of expectations leading to convergence in the dynamic steady state.

The third is *rational inattention*, which examines the possibility of information-gathering being costly (at the very least in terms of time), and therefore leads to endogenous acquisition of information. Some of the work on this Sims (2005), Sims (2011), Paciello (2012), Adam (2007) and Luo and Young (2009) leads to an analysis that depends on the choice of the variance of shocks based on II in a representative agent model. Others, notably Mackowiak and Wiederholt (2009), Mackowiak and Wiederholt (2011) are investigated within the context of II with heterogenous agents. The background to this is discussed in Section 6, where we show that if idiosyncratic shocks have negligible variance compared with aggregate shocks, then the aggregate economy under II has the same behaviour as II in a representative agent model; this our rationale for the remainder of the paper.

3 The ABC and D of VARs

Before addressing the main issues associated with rational expectations, we summarize the main results of the seminal paper Fernandez-Villaverde *et al.* (2007) associated with the reduced form of a linear stochastic VARMA model. Suppose the system is written in the form in the form:

$$v_t = \tilde{A}v_{t-1} + \tilde{B}\epsilon_t, \qquad \epsilon_t \sim N(0, I) \tag{1}$$

$$m_t^E = \tilde{G}v_t \equiv \tilde{C}v_{t-1} + \tilde{D}\epsilon_t, \quad \tilde{C} \equiv \tilde{G}\tilde{A}, \quad \tilde{D} \equiv \tilde{G}\tilde{B} \tag{2}$$

where m_t^E are the observables for the econometrician[3] and the system matrix $\tilde{A}$ has stable eigenvalues. The use of $m_t^E = \tilde{G}v_t$ is not the exact form used by Fernandez-Villaverde *et al.* (2007), but is utilised because it reduces all the subsequent notation.

[3]Later we distinguish between the observables of the econometrician and those of the economic agents.

3.1 The Invertibility Conditions

The system is called invertible if the values of the shocks ε_t can be recovered from the observations m_t^E; in lag operator form it is clear that $\varepsilon_t = (\tilde{G}(I - \tilde{A}L)^{-1}\tilde{B})^{-1}m_t^E$. Naturally enough we require A to be a stable matrix, but the first main result[4] of Fernandez-Villaverde *et al.* (2007) is

Condition 1 (System Invertibility Conditions). *(1) $\tilde{G}\tilde{B}$ is square and invertible; (2) $\tilde{A} - \tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}\tilde{A}$ and $\tilde{A} - \tilde{A}\tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}$ have stable eigenvalues, with each implying the stability of the other.*

We shall focus on the second of these matrices shortly. The latter condition arises from deriving an expression for ε_t in terms of current and past values of m_t^E:

$$\epsilon_t = (\tilde{G}\tilde{B})^{-1}m_t^E - (\tilde{G}\tilde{B})^{-1}\tilde{G}\tilde{A}\sum_{j=1}^{\infty}(\tilde{A} - \tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}\tilde{A})^j\tilde{B}(\tilde{G}\tilde{B})^{-1}m_{t-j}^E \tag{3}$$

and noting that $\tilde{A}(\tilde{A} - \tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}\tilde{A})^j = (\tilde{A} - \tilde{A}\tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G})^j\tilde{A}$.

If $\tilde{A} - \tilde{A}\tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}$ does not have stable eigenvalues, then m_t^E is still a square-summable stochastic process, and there exists an *innovations representation* that does not converge to the shocks ε_t. The simplest example of this is the MA(1) process $m_t^E = \varepsilon_t + \theta\varepsilon_{t-1}$. If $|\theta| < 1$, then the system is invertible; if $|\theta| > 1$, then observation of m_t^E over time will yield an innovations process $\hat{\varepsilon}_t$ that satisfies $m_t^E = \hat{\varepsilon}_t + 1/\theta\hat{\varepsilon}_{t-1}$. Fernandez-Villaverde *et al.* (2007) provide an example for observations of the output/consumption surplus in a simple permanent income consumption model.

3.2 The Innovations Representation

The innovations representation is closely associated with the Kalman filtering solution to the signal processing problem posed by (1). The Riccati equation associated with this is given in steady state by

$$\tilde{P} = \tilde{A}\tilde{P}\tilde{A}' - \tilde{A}\tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\tilde{G}\tilde{P}\tilde{A}' + \tilde{B}\tilde{B}' \tag{4}$$

[4] This appears to date back at least to the work of Brockett and Mesarovic (1965).

The relevant (unique) solution for $\tilde{P}$ is positive definite and must satisfy the requirement that $K = \tilde{A} - \tilde{A}\tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\tilde{G}$ is a stable matrix. The reason why this is a relevant condition is because with this definition of K, we can rewrite (4) as $\tilde{P} = K\tilde{P}K' + \tilde{B}\tilde{B}'$. If we reframe this as an updating equation for $\tilde{P}$ i.e. $\tilde{P}_{t+1} = K_t\tilde{P}_tK_t' + \tilde{B}\tilde{B}'$, it is clear that when K has eigenvalues outside the unit circle, any small deviations of $\tilde{P}_t$ from its corresponding steady state will immediately lead to local instability.

A little manipulation of the results of Anderson and Moore (1979), Chapter 9, enables us to obtain the innovations representation and the innovations process $\hat{\epsilon}_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$

$$\bar{v}_{t+1} = \tilde{A}\bar{v}_t + \tilde{A}\tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\hat{\varepsilon}_{t+1}, \qquad \hat{\varepsilon}_t = m_t^E - \tilde{G}\bar{v}_t \qquad \hat{\varepsilon}_t \sim N(0, \tilde{G}\tilde{P}\tilde{G}') \tag{5}$$

This representation corresponds to the filtering problem as $t \to \infty$, and is used below when we examine the estimation of a model under imperfect information; $\bar{v}_{t+1}$ represents the best estimate of v_{t+1} using the information set comprising all observations of $m_s^E,\ s \leq t$. An alternative representation gives a match to (1); define $\tilde{v}_t = \bar{v}_{t-1} + \tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\hat{\varepsilon}_t$. Then can then rewrite (5) as

$$\tilde{v}_t = \tilde{A}\tilde{v}_{t-1} + \tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\hat{\varepsilon}_t \qquad m_t^E = \tilde{G}\tilde{v}_t \qquad \hat{\varepsilon}_t \sim N(0, \tilde{G}\tilde{P}\tilde{G}') \tag{6}$$

When the system is invertible, a potential candidate Riccati solution is $\tilde{P} = \tilde{B}\tilde{B}'$ subject to the stability of $\tilde{A} - \tilde{A}\tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\tilde{G} = \tilde{A} - \tilde{A}\tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}$, which is exactly the requirement that we encountered before considering the filtering issue. In this case the dynamic equation in (6) becomes

$$\tilde{v}_t = \tilde{A}\tilde{v}_{t-1} + \tilde{B}(\tilde{G}\tilde{B})^{-1}\hat{\varepsilon}_t \qquad \hat{\varepsilon}_t \sim N(0, \tilde{G}\tilde{B}\tilde{B}'\tilde{G}') \tag{7}$$

Defining $\varepsilon_t = (\tilde{G}\tilde{B})^{-1}\hat{\varepsilon}_t$ transforms (7) into (1).

The main relevance of this result is when the relationship between the variables is estimated as a finite VAR approximation to the VARMA; only for the case $\tilde{A} - \tilde{A}\tilde{B}(\tilde{G}\tilde{B})^{-1}\tilde{G}$ and $\tilde{A}$ having stable eigenvalues can the shocks ϵ_t be approximately recovered. If the invertibility conditions are not satisfied, then the solution for $\tilde{P}$ is no longer $\tilde{P} = \tilde{B}\tilde{B}'$, and the VARMA approximation will generate a series of residuals that correspond to $\hat{\varepsilon}_t$

in (6).

The innovations representation is closely connected to the use of the Kalman filter in the estimation of linear models. Suppose that the system is given by (1). Then the loglikelihood lnL for the system is given by

$$2lnL = -Trln(2\pi) - \sum_{t=1}^{T}[\ln\det(\mathrm{cov}(\hat{\epsilon}_t)) + \hat{\epsilon}_t'(\mathrm{cov}(\hat{\epsilon}_t))^{-1}\hat{\epsilon}_t] \quad (8)$$

where the innovations process $\hat{\epsilon}_t \equiv m_t^E - \mathbb{E}_{t-1}m_t^E$, T is the number of time periods and r is the dimension of m_t^E. In this case we use the time-varying versions of the Riccati equation (4) and of the state updating equations (10) to generate the values of $\hat{\epsilon}_t$ and $\mathrm{cov}(\hat{\epsilon}_t)$:

$$\tilde{P}_{t+1} = \tilde{A}\tilde{P}_t\tilde{A}' - \tilde{A}\tilde{P}_t\tilde{G}'(\tilde{G}\tilde{P}_t\tilde{G}')^{-1}\tilde{G}\tilde{P}_t\tilde{A}' + \tilde{B}\tilde{B}' \quad (9)$$

$$\bar{v}_{t+1} = \tilde{A}\bar{v}_t + \tilde{A}\tilde{P}_t\tilde{G}'(\tilde{G}\tilde{P}_t\tilde{G}')^{-1}\hat{\varepsilon}_t, \qquad \hat{\varepsilon}_t = m_t^E - \tilde{G}\bar{v}_t \qquad \hat{\varepsilon}_t \sim N(0, \tilde{G}\tilde{P}_t\tilde{G}') \quad (10)$$

4 Invertibility under Perfect and Imperfect Information

We write a linear rational expectations (RE) model in the following general form:

$$A_0Y_{t+1,t} + A_1Y_t = A_2Y_{t-1} + \Psi\varepsilon_t \qquad m_t^E = L^EY_t \qquad m_t^A = L^AY_t \quad (11)$$

with $Y_{s,t}$ is shorthand for expectations of Y_s using information available at time t now referring to expectations subject to an information sets available to economic agents and the econometrician $I_t = \{m_k : k \leq t\}$ that may be imperfect. Under imperfect information we can introduce an additional term involving $Y_{t,t}$, the expected current value of Y_t by defining a new variable $Y_t^1 = Y_{t-1}$, and then represent $Y_{t,t}$ by $Y_{t+1,t}^1$.

For perfect information $L^A = I$, the identity matrix. For the case of imperfect information in what follows we assume $m_t^A = m_t^E$ is also the vector of observables from the previous section available to the econometrician.[5]

We assume that all agents have the same information set. Note that measurement

[5] Our solution procedure can be generalized to allow for different imperfect information observables $m_t^A \neq m_t^E$ - see Corollary 2 of Theorem 3. A more general case that allows for the various forward expectations each to be formed by differently informed agents is studied in Lubik *et al.* (2017) who show that this results in different Blanchard-Kahn stability conditions.

errors can be accounted for by including them in the vector ε_t.

4.1 Solution under Perfect and Imperfect Information

Anderson (2008) lists a selection of methods that can be used to solve (11) for the case when agents have perfect information. The most well-known of these are Sims (2002) and Blanchard and Kahn (1980) - henceforth B-K - but as the former points out, it is not always obvious how to write a system of the form (11) in B-K form.

For our imperfect information case, we shall be using a version of the B-K form that was utilised by Pearlman *et al.* (1986), which provided a solution under imperfect information. In order to move seamlessly from (11) to results that are based on Pearlman *et al.* (1986), it is necessary to show that (11) can be converted into the state-space representation

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} \tag{12}$$

$$m_t^E = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{13}$$

where z_t, x_t are vectors of backward and forward-looking variables, respectively, and ϵ_t is a vector of shock variables. Details of this derivation are in Appendix B. For later convenience we define

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{14}$$

The reason for transforming the equations of the model from (11) is that the corresponding solution method of Sims (2002) does not extend easily to imperfect information; we give an indication of how to modify the latter method to extend to imperfect information in Appendix C, and the implication is that to complete the task requires techniques no less complicated than those used in Pearlman *et al.* (1986).

Theorem 1. *(a) Under perfect information,* (11) *can always be converted into B-K form; (b) Under imperfect information, one can obtain the B-K form, but it may require invoking the expectational saddlepath relationship* $x_{t+1,t} + Nz_{t+1,t} = 0$.

The proof of Theorem 1(a) is in Appendix B. To prove (b) we present an example that shows that one needs to invoke the expectational saddlepath to obtain the B-K form:

Proof of Theorem 1(b).

$$x_{t+1,t} + \alpha x_t + s_t = \beta s_{t-1} + \epsilon_t \qquad x_t - x_{t,t} + s_{t,t} = \gamma s_{t-1} \tag{15}$$

It is clear from examining these equations that they cannot be manipulated into B-K form directly. However, if we now advance these equations by k periods and take expectations subject to I_t, one obtains two equations relating $x_{t+k+1,t}, s_{t+k,t}$ to $x_{t+k,t}, s_{t+k-1,t}$. Since this is true for all $k \geq 1$, and provided there is exactly one unstable eigenvalue corresponding to these dynamic relationships, it follows that there must be an expectational saddlepath relationship $x_{t+1,t} = -ns_{t,t}$. Substituting this into the first of the above equations allows us to solve in particular for s_t in terms of $x_t, s_{t,t}, s_{t-1}, \epsilon_t$; from the second equation we can solve for $s_{t,t}$ in terms of $s_{t-1,t}$, so that we can replace the second equation by an equation for s_t in terms of $x_t, s_{t-1,t}, s_{t-1}, \epsilon_t$. Redefining $z_{t+1} = s_t$, it is now straightforward to obtain the B-K form for the first equation and the new second equation. ■

The expressions involving $z_{t,t}, x_{t,t}$ arise from rewriting the model in B-K form (12). The presence of these terms is what distinguishes our results on invertibility from those of Baxter *et al.* (2011), and in addition - as pointed out in the introduction - we do not make the assumption that agents have full current information on all variables for which forward expectations are present in the model. A full derivation of the solution for the general linear setup above is provided in Pearlman *et al.* (1986), but is outlined below.

4.1.1 Perfect Information Case

We first consider the solution for (12) and (13) under *perfect information*; in this case we assume that all stocks dated $t-1$ and other variables dated t in (12) are fully observed during the course of period t. These would include for example beginning-of-period capital stock, beginning-of-period net worth in banking models, all flows such as output, consumption, investment, all output and factor prices, inflation over the period and all end-of-period realizations of exogenous stochastic processes.

For this perfect information case (where $z_{t,t} = z_t, x_{t,t} = x_t$) there is a saddle path satisfying:

$$x_t + Nz_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G+H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{16}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G+H)$ is the same as the dimension of x_t, then the system will be determinate.[6] (13) now becomes:

$$m_t^E = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{17}$$

$$m_t^A = \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{18}$$

We then ask whether *limited* observations by the econometrician of the form (17) in B-K form will lead to invertibility. These observations are limited in that the full state vector (including for instance the shock processes) is not part of the econometrician's data set.

From the saddle path relationship (16), it is clear that the reduced-form representation of the model is now

$$z_t = (G_{11} + H_{11} - (G_{12} + H_{12})N)z_{t-1} + B\epsilon_t \equiv Az_{t-1} + B\epsilon_t \tag{19}$$

$$x_t = -Nz_t = -NAz_{t-1} - NB\epsilon_t \tag{20}$$

$$m_t^E = (M_1 + M_3 - (M_2 + M_4)N)z_t \equiv Ez_t \tag{21}$$

$$m_t^A = \begin{bmatrix} z_t \\ x_t \end{bmatrix} = \begin{bmatrix} I \\ -N \end{bmatrix} z_t \tag{22}$$

We now note that (19) – (21) together are of the form of (1). However, in order to reduce excess notation both here and subsequently, and because the dynamics of the forward-looking variables x_t are not relevant to the main results, where necessary we shall just refer to the relationship between x_t and other current variables, namely $x_t = -Nz_t$.

Thus from Fernandez-Villaverde *et al.* (2007) we deduce that a necessary and sufficient condition for invertibility is that (i) $\tilde{D} = EB$ is invertible (ii) $A - AB(EB)^{-1}E$ and A

[6] Note that in general the dimension of x_t will not match the number of expectational variables in (11). The algorithm in Appendix B will eliminate linear dependency among expectational variables and will also convert the system $a_t = \rho a_{t-1} + \varepsilon_t, \;\; b_t = \mathbb{E}_t a_{t+1}$ into $a_t = \rho a_{t-1} + \varepsilon_t, \;\; b_t = \rho a_{t,t}$.

have stable roots (using the definitions of A and E in (19) and (21)).

More generally, from (4) and (6) we note that the innovations representation under perfect information for agents and observations m_t^E for econometricians is given by

$$\tilde{v}_t = A\tilde{v}_{t-1} + SE'(ESE')^{-1}\hat{\epsilon}_t \qquad m_t^E = E\tilde{v}_t \qquad \hat{\epsilon}_t \sim N(0, ESE') \tag{23}$$

where S satisfies the Riccati equation

$$S = ASA' - ASE'(ESE')^{-1}ESA' + BB' \tag{24}$$

4.1.2 Imperfect Information Case

We first briefly outline how the imperfect information setup is solved, and the provide the conditions for invertibility. Following Pearlman *et al.* (1986), we use the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + J \left[m_t - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right]$$

The Kalman filter was developed in the context of backward-looking models, but extends as we see here to forward-looking models. The basic idea behind it is that the best estimate of the states $\{z_t, x_t\}$ based on current information is a weighted average of the best estimate using last period's information and the new information m_t. Thus the best estimator of the state vector at time $t-1$ is updated by multiple J of the error in the predicted value of the measurement as above, where J is given by

$$J = \begin{bmatrix} PD' \\ -NPD' \end{bmatrix} \Gamma^{-1}$$

and $D \equiv M_1 - M_2 G_{22}^{-1} G_{21}$, $\Gamma \equiv (M_1 - M_2 N)PD'$ where P satisfies the Riccati equation (28) below.

Using the Kalman filter, the solution as derived by Pearlman *et al.* (1986) [7] is given by the following processes describing the pre-determined and non-predetermined variables

[7] A less general solution procedure for linear models with imperfect information is in Lungu *et al.* (2008) with an application to a small open economy model, which they also extend to a non-linear version.

$z_t = \tilde{z}_t + z_{t,t-1}$ and x_t, and a process describing the innovations $\tilde{z}_t \equiv z_t - z_{t,t-1}$:

$$\begin{aligned}
\text{Predetermined}: \quad z_{t+1,t} &= Az_{t,t-1} + APD'(DPD')^{-1}D\tilde{z}_t && (25)\\
\text{Non-predetermined}: \quad x_t &= -Nz_{t,t-1} - G_{22}^{-1}G_{21}\tilde{z}_t - (N - G_{22}^{-1}G_{21})PD'(DPD')^{-1}D\tilde{z}_t\\
\text{Innovations}: \quad \tilde{z}_t &= F[I - PD'(DPD')^{-1}D]\tilde{z}_{t-1} + B\epsilon_t && (26)\\
\text{Measurement Equation}: \quad m_t &= Ez_{t,t-1} + EPD'(DPD')^{-1}D\tilde{z}_t && (27)
\end{aligned}$$

where[8]

$$A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N, \quad F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21}, \quad D \equiv M_1 - M_2G_{22}^{-1}G_{21}$$

and P is the solution of the Riccati equation given by

$$P = FPF' - FPD'(DPD')^{-1}DPF' + BB' \tag{28}$$

where we assume that the shocks are normalized such that their covariance matrix is given by the identity matrix. Corresponding to the relevant solution for P, we also need it, as discussed earlier, to satisfy the convergence condition, that $F - FPD'(DPD')^{-1}D$ has all eigenvalues in the unit circle.

We can see that the solution procedure above is a generalization of the B-K solution for perfect information and that the determinacy of the system is independent of the information set. We also note that (25), (26) and (27) together are of the form (1)[9]:

$$\begin{aligned}
v_t &= \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} = \tilde{A}v_{t-1} + \tilde{B}\epsilon_t && (29)\\
m_t &= \tilde{G}v_t && (30)
\end{aligned}$$

[8] Fz_t and Dz_t respectively represent those parts of the dynamics of z_t and of the measurement m_t that do not depend on expectations. This arises from eliminating x_t from each of the latter equations.

[9] We could include x_t in the array v_t, but since it is only dependent on current values of $\tilde{z}_t$ and $z_{t,t-1}$, this just adds to notation without altering any results.

where

$$\tilde{A} = \begin{bmatrix} A & APD'(DPD')^{-1}D \\ 0 & F[I - PD'(DPD')^{-1}D \end{bmatrix} \tag{31}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{32}$$

$$\tilde{G} = [E \quad EPD'(DPD')^{-1}D] \tag{33}$$

4.2 Invertibility and Agents' Information Sets

The conventional approach implicit in the standard Bayesian estimation procedure is that all agents have perfect information about variables and shocks, whereas the econometrician has only a limited (imperfect) information set. Here by contrast we examine the case when agents and econometricians have the same imperfect information set and we ask the following question that compares invertibility for the two scenarios. Suppose agents have perfect information and the information set available to the econometrician is such that the process is invertible; if the agents have the same information set as the econometrician, is the process still invertible?

Fernandez-Villaverde *et al.* (2007) do not attempt to address this question. Their focus is on the complete reduced form of the solution from the perspective of the econometrician; the source of this reduced form i.e its dependence on the information set of the agents is investigated. As we have seen above, the reduced form under any information set is of the standard state space type investigated by Fernandez-Villaverde *et al.* (2007), but the invertibility properties depend on the information set. We now present the first main theoretical result of the paper.

Theorem 2. *Assume for the given information set of the econometrician, that the system is invertible under agents' full information (i.e. as shown at the end of the last subsection, EB is invertible). For the imperfect information solution to be (i) equivalent to the full information solution when the number of observables is equal to the number of shocks, and (ii) invertible, the necessary and sufficient conditions are that the square matrix DB is of full rank, and* $F(I - B(DB)^{-1}D)$ *and* $A(I - B(EB)^{-1}E)$ *are stable matrices.*

Proof of Theorem 2. For (i) sufficiency results from the solution to (28) being $P =$

BB'; it is easy to check this, since $PD'(DPD')^{-1}DP = BB'D'(B'D')^{-1}(DB)^{-1}DBB' = BB'$. In addition P must satisfy the Convergence Condition that $F - FPD'(DPD')^{-1}D$ a stable matrix. The latter in this case is equal to $F(I - B(DB)^{-1}D)$, in which case $\tilde{z}_t = B\epsilon_t + (F(I - B(DB)^{-1}D))^t\tilde{z}_0$, which in dynamic equilibrium implies $\tilde{z}_t = B\epsilon_t$. This implies that $z_{t+1,t} = Az_{t,t-1} + AB\epsilon_t$, and hence that $z_{t+1} = \tilde{z}_{t+1} + z_{t+1,t} = Az_{t,t-1} + AB\epsilon_t + B\epsilon_{t+1} = Az_t + B\epsilon_{t+1}$ as in the perfect information case. In addition, from (27), $m_t = Ez_{t,t-1} + E\tilde{z}_t = Ez_t$, also as in the perfect information case.

Necessity of (i) follows from the fact that if DB is not of full rank, then $P = BB'$ cannot be a solution to (28), because then $(DPD')^{-1}$ would not exist.

For (ii), substitute for $z_{t,t-1}$ and $\tilde{z}_t$ from (25) and (26), into the expression for m_t, so that the latter is written in terms of lagged states and current shocks; the last term in the expression is $EPD'(DPD')^{-1}DB\epsilon_t$. From Fernandez-Villaverde *et al.* (2007), invertibility requires $EPD'(DPD')^{-1}DB$ to be of full rank, so DB must be full rank. From the proof of (i), it follows in that case that $EPD'(DPD')^{-1}DB = EB$, and EB is full rank for the perfect information setup to be invertible.

Finally, the requirement that $A(I - B(EB)^{-1}E)$ be stable is exactly as in the perfect information case. ■

From the authors' experience with numerous RE models, the most common non-obvious reason for non-invertibility is associated with:

Corollary 1. *If EB is of full rank (i.e. number of observables = number of shocks) and invertible, but D is not of full row rank, then imperfect information is not equivalent to full information, and the system is then not invertible.*

Proof of Corollary 1. Writing (27) in terms of lagged state variables and shocks yields a coefficient matrix on the latter given by $EPD'(DPD')^{-1}DB$, and the rank of this is $\leq rank(DB) \leq rank(D)$. This immediately implies that the system is non-invertible. ■

These are new results in the literature, namely that if a econometrician's limited information set under perfect information is invertible, it does not follow that the same limited information set under imperfect information is also invertible. The reason for this is as follows. Suppose that $D = M_1 - M_2G_{22}^{-1}G_{21}$ is not of full rank, and that U is a matrix that satisfies $UD = 0$ i.e. $UM_1 = UM_2G_{22}^{-1}G_{21}$. Define V as the orthogonal complement of U, and rewrite the set of measurements m_t as their linear transformation $m_t^U = Um_t$ and $m_t^V = Vm_t$. Then in particular

$$
\begin{aligned}
m_t^U &= \begin{bmatrix} UM_1 & UM_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \\
&= \begin{bmatrix} UM_2 G_{22}^{-1} G_{21} & UM_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \\
&= UM_2 G_{22}^{-1} \left(\mathbb{E}_t x_{t+1} - \begin{bmatrix} H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \right) + \begin{bmatrix} UM_3 & UM_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix}
\end{aligned}
\tag{34}
$$

where the last expression comes from substituting from (12). Noting that $\mathbb{E}_t x_{t+1} = -N\mathbb{E}_t z_{t+1}$, and that $\mathbb{E}_t z_{t+1}$ is dependent on $\mathbb{E}_t z_t$ and $\mathbb{E}_t x_t$, it follows that m_t^U is solely dependent on these too. In other words, m_t^U cannot be affected by current shocks ϵ_t, and is redundant information.

Note too, that if EB is invertible, but DB is not invertible despite D being of full rank, then this is implies that the imperfect information set in effect contains a lagged variable.

4.3 The Innovations Process under Imperfect Information

We now examine the innovations process under imperfect information as in (10) and (6).

Theorem 3. *If the system under imperfect information is not invertible, then the VARMA process that generates the impulse response functions of the fundamental shocks is of a higher order than the VARMA innovations representation of the system* ***which is of the same order as for perfect information****.*

Proof of Theorem 3. We first solve the Riccati equation corresponding to the matrices (31)-(33). It is easy to verify that $\tilde{P} = diag(M, P)$ where $M = Z - PD'(DPD')^{-1}DP$ and Z satisfies

$$Z = AZA' - AZE'(EZE')^{-1}EZA' + PD'(DPD')^{-1}DP \tag{35}$$

After further substitution, we can show that the innovations representation corresponding to (10) is given by

$$\bar{v}_{t+1} = \begin{bmatrix} A & APD'(DPD')^{-1}D \\ 0 & F - FPD'(DPD')^{-1}D \end{bmatrix} \bar{v}_t + \begin{bmatrix} FZE'(EZE')^{-1} \\ 0 \end{bmatrix} \hat{\varepsilon}_t \qquad \hat{\varepsilon}_t = m_t^E - \tilde{G}\bar{v}_t \tag{36}$$

or more succinctly

$$\bar{v}_{1,t+1} = A\bar{v}_{1t} + AZE'(EZE')^{-1}\hat{\varepsilon}_t \qquad \hat{\varepsilon}_t = m_t^E - E\bar{v}_{1t} \tag{37}$$

The corresponding VARMA representation arises from defining $\tilde{v}_{1t} = \bar{v}_{1t} + ZE'(EZE')^{-1}\hat{\varepsilon}_t$, which yields

$$\tilde{v}_{1,t+1} = A\tilde{v}_{1t} + ZE'(EZE')^{-1}\hat{\varepsilon}_{t+1} \qquad m_t^E = E\tilde{v}_{1t} \tag{38}$$

The final step follows from comparing (38) with (25) and (26); clearly the dynamics of the system explained by the innovations process $\hat{\varepsilon}_t$ are of smaller dimension that the dynamics yielding the impulse responses. ■

Remarkably, this result tells us that even though the dynamics of the system under imperfect information are considerably more complex and add more inertia than under perfect information, the innovations process is generated by equations that are of the same dimension in each case. What this means is that the statistical properties of data as generated by the model under imperfect information and as estimated by a standard VAR (or VARMA) cannot in general generate the IRFs that one would obtain by estimating the parameters of a DSGE model under imperfect information.

A criticism of the imperfect information approach that we have been using thus far is that it is likely that agents will have more information about the variables of the model than the econometrician has, although this does not necessarily imply that agents have perfect information. In particular they are unlikely to be fully informed about the current values of all the AR(1) shocks in the model. This would imply that the properties of the model solution still embody those of Theorem 3, i.e., that a VAR estimation by agents would not be able to replicate the IRFs of shocks.

Corollary 2. *If the econometrician's information set is a subset of that of the agents,* *then the innovations process as estimated by the econometrician will be of a lower order than that of the IRFs of the fundamental shocks* ***and will again be of the same order as for perfect information****.*

Proof of Corollary 2. The state space equations describing the system, (25), (26), will be unchanged, as these depend on the measurements made by the agents. However if the information set of the econometrician is a subset of that of the agents, this means that in the notation of (11), we have $L^E = WL^A$ for some matrix W. It then follows that the measurement equation of the econometrician, following from (27), is given by $m_t = W(Ez_{t,t-1} + EPD'(DPD')^{-1}D\tilde{z}_t)$. Thus the innovations process and the VARMA as shown in the proof of Theorem 3 are changed merely by replacing E by WE, with the Riccati matrix Z also obtained with the same replacement of E. ■

The implication therefore is that if agents have any form of imperfect information, then provided the econometrician is no better informed than the agents, then an unrestricted VAR (or indeed VARMA) cannot possibly be used to generate the true IRFs of the shocks.[10]

Example 1. Consider a New Keynesian Phillips curve dependent on marginal cost m_t and a mark-up shock $\epsilon_{1,t}$ *assumed exogenous*

$$\pi_t = \beta(\pi_{t+1,t} + m_t + \epsilon_{1,t}) \tag{39}$$

$$m_{t+1} = \rho m_t + \epsilon_{2,t+1} \tag{40}$$

where $\epsilon_{i,t} \sim N(0, \sigma_i^2)$ which is of the Blanchard-Kahn state-space form:

$$\begin{bmatrix} \epsilon_{1,t+1} \\ m_{t+1} \\ \mathbb{E}_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ -1 & -1 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ m_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \\ 0 \end{bmatrix}$$

[10] Theorem 3 is closely related to Baxter *et al.* (2011), Corollary 1, p302: they write "the behavior of the estimated states and forward-looking variables is isomorphic to the behavior of the true states under full information,". What they mean by this is that the innovations representation is the same except for the covariance matrix, which is our Theorem 3. But our result is more general because unlike theirs it allows for forward-looking variables not necessarily all to be observed (e.g. with Calvo pricing, optimal price is set, but inflation might be observed with measurement error). Our result is significantly different in that in their paper, the forward looking variables are just a linear function of expectations of the backward looking variables (because they are fully observed), whereas in our case the representation is more complicated. We are grateful to Stephen Wright for drawing our attention to this similarity.

Now consider **information sets**:

- Perfect Information (PI) : $[\epsilon_{1,t}\ m_t\ \pi_t]'$ with no measurement error.
- Imperfect Information (II): π_t with no measurement error

Consider first the solution under PI. From our general solution procedure above, the following matrices are defined

$$A = F = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}; \quad N = -E = \begin{bmatrix} \beta & \dfrac{\beta}{1-\beta\rho} \end{bmatrix}; \quad D = [\beta\ \beta]; \quad BB' = \begin{bmatrix} \sigma^2_{\varepsilon_1} & 0 \\ 0 & \sigma^2_{\varepsilon_2} \end{bmatrix}$$

It follows that under PI that

$$\pi_t = \beta\epsilon_{1,t} + \frac{\beta}{1-\beta\rho} m_t \tag{41}$$

This is an **VAR(1)** process in $[\pi_t\ m_t]'$ and $[\epsilon_{1,t}\ \epsilon_{2,t}]'$.

Now compare with the solution where agents have II.

$$\begin{aligned} m_t &= \rho m_{t-1} + \epsilon_{2,t} \\ \tilde{m}_t &\equiv m_t - m_{t,t-1} = \frac{\rho}{\sigma_1^2 + p}(\sigma_1^2 \tilde{m}_{t-1} - p\epsilon_{1,t-1}) + \epsilon_{2,t} \end{aligned} \tag{42}$$

$$\begin{aligned} \pi_t &= \beta\left(1 + \frac{\beta\rho p}{(1-\beta\rho)(\sigma_1^2 + p)}\right)\epsilon_{1,t} + \frac{\beta}{1-\beta\rho} m_t \\ &- \frac{\beta^2\rho\sigma_1^2}{(1-\beta\rho)(\sigma_1^2+p)}\tilde{m}_t \end{aligned} \tag{43}$$

where the steady-state Ricatti equation is:

$$P = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & p \end{bmatrix} \qquad \text{where } p = \frac{\rho^2 p \sigma_1^2}{\sigma_1^2 + p} + \sigma_2^2$$

noting that $N - G_{22}^{-1}G_{21} = \begin{bmatrix} 0 & \frac{\beta^2\rho}{1-\beta\rho} \end{bmatrix}$, This is an **VARMA(1,1)** process in $[\pi_t\ m_t\ \tilde{m}_t]'$ and $[\epsilon_{1,t}\ \epsilon_{2,t}]'$.

Figure 1 shows the impulse response function following a negative marginal cost shock $\epsilon_{2,t}$. The greater is σ_1^2, the greater is the difference between II and PI.

To obtain the innovations representation, we first recall that $\bar{v}_t$ represent the best

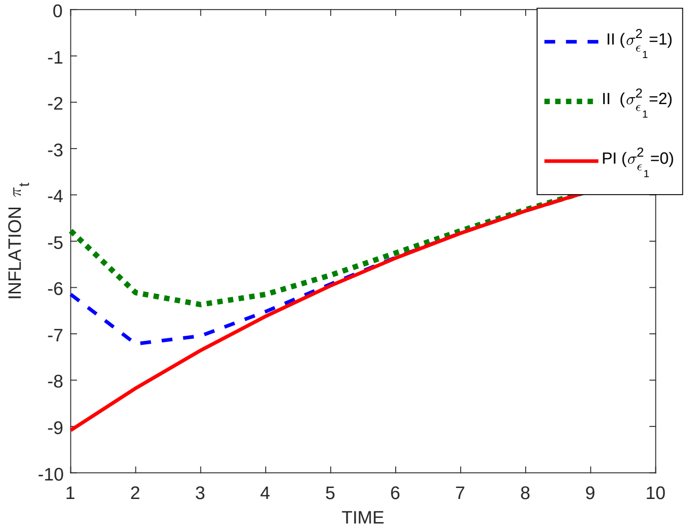

Figure 1: Inflation Dynamics under Perfect (PI) and Imperfect Information (II)

estimate of v_t using the information set comprising all observations of m_s^E, $s \leq t-1$ and we define $\tilde{v}_t = \bar{v}_{t-1} + \tilde{P}\tilde{G}'(\tilde{G}\tilde{P}\tilde{G}')^{-1}\hat{\varepsilon}_t$. Note that for this example EPD' is just a scalar, which is greater than 0 because $\frac{1}{\beta} - \rho > 1 - \rho > 0$. Hence it follows that the solution for Z in (35) is given by

$$Z = PD'(DPD')^{-1}DP = \frac{1}{\sigma_w^2 + p}\begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix}\begin{bmatrix} \sigma_1^2 & p \end{bmatrix} \tag{44}$$

The innovations process that provides the VARMA for π_t, corresponding to (38) is then

$$\begin{aligned} \tilde{v}_{1,t} &= \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix}\tilde{v}_{1,t-1} + \frac{1}{\beta\sigma_1^2 + \frac{\beta}{1-\beta\rho}p}\begin{bmatrix} \sigma_1^2 \\ p \end{bmatrix}\hat{\epsilon}_{2,t} \\ \pi_t &= \begin{bmatrix} \beta & \dfrac{\beta}{1-\beta\rho} \end{bmatrix}\tilde{v}_{1,t} \end{aligned}$$

from which it is readily seen that the system is back to a **VAR(1)** process as under PI. This illustrates **Theorem 3**: even though II adds more persistence than under PI, the innovations process dynamics has the same dimensions in each case.

This seemingly puzzling result is not one that is confined to imperfect information, but is prevalent in many general time series representations. Consider therefore the following stationary process, expressed in lag-operator form:

$$y_t = \frac{1}{1-\alpha L}\left(\frac{k}{1-\beta L}\varepsilon_{1t} + \varepsilon_{2t}\right) \qquad \varepsilon_{it} \sim N(0, \sigma_i^2),\ i = 1, 2 \tag{45}$$

where $\varepsilon_{1t},\ \varepsilon_{2t}$ are independent of one another. The spectrum $S_y(\omega)$ of this process is given by

$$S_y(\omega) = \frac{1}{(1-\alpha L)(1-\alpha L^{-1})}\left(\frac{k^2\sigma_1^2}{(1-\beta L)(1-\beta L^{-1})} + \sigma_2^2\right) \tag{46}$$

where $L = e^{i\omega}$. It is easy to show that the spectrum is proportional to $1/((1-\beta L)(1-\beta L^{-1}))$ when $k^2\sigma_1^2 = \sigma_2^2(\beta-\alpha)(\frac{1}{\alpha\beta}-1)$ i.e. for a particular value of k, y_t is no longer an ARMA(2,1) process, but is merely an AR(1) process. Thus in the analysis above of the innovations process, the matrix P plays the role of k in this latter equation, and the parameters of the VARMA are nonlinearly dependent on the variances of the shocks.

4.3.1 Implications for Estimation under Imperfect Information

We now examine what this implies firstly for VAR estimation under imperfect information, and secondly show that that under imperfect information a singularity problem can potentially exist.

With regard to VAR estimation, we know that DSGE models will usually have far fewer parameters than the corresponding VAR, so that a VAR that fully takes account of the DSGE model would require cross-equation restrictions. For the DSGE model under imperfect information to be similarly estimated as a VAR would require cross-equation restrictions that not only involve the underlying parameters, but also involve the variances of the shocks. This is why a standard VAR estimation that merely reproduced the innovations process might be unsuitable for assessing the individual effects of the fundamental shocks under imperfect information. The reason, according to Theorem 3, is that the innovations process is a lower order VARMA than the VARMA that corresponds to each of the individual shocks.

Turning to the estimation of these models assuming the same information set for agents as for econometricians, as in the case of Section 3, we use a time varying version of (37) and of the Riccati matrix in order to evaluate the loglikelihood (8) for any given set of parameters:

$$\bar{v}_{t+1} = A\bar{v}_t + AZ_tE'(EZ_tE')^{-1}\hat{\epsilon}_t \qquad \hat{\epsilon}_t \equiv m_t^E - E\bar{v}_t$$

$$Z_{t+1} = AZ_tA' - AZ_tE'(EZ_tE')^{-1}EZ_tA' + PD'(DPD')^{-1}DP \tag{47}$$

Initial values are $\bar{v}_1 = 0$, with Z_1 satisfying $Z_1 = AZ_1A' + PD'(DPD')^{-1}DP$.[11]

We now briefly focus on a problem that could potentially arise with estimation under imperfect information.

Recall what is meant by over-identification, or the singularity problem in estimation. As Canova *et al.* (2014) point out, if the number of observables exceeds the number of shocks, then the likelihood function will be singular[12]. We then obtain a further corollary to Theorem 3:

Corollary 3. *If rank*$(D) <$ *the number of observables, then the system under imperfect information is singular.*

Proof of Corollary 3. If D is of full row rank, then it is easy to see that in general $PD'(DPD')^{-1}DP$ will have the same rank as D. If D is not of full row rank, then rank$(PD'(DPD')^{-1}DP)$ $\leq$rank(D) i.e. the 'effective' number of shocks is less than the number of observables. If D is not of full rank, then we can solve for P by writing $D = UD_1$, where D_1 has a smaller number of rows than D, and is of full row rank, and $U'U = I$. Then an appropriate likelihood function is obtained by changing the observables from m_t to $U'm_t$. An alternative of course is to incorporate measurement error into the system. ■

Note that these results are only relevant when the measurements satisfy invertibility if agents were to have perfect information. If any of the measurements are lagged, then Theorem 3 does not apply.

[11] It is inappropriate for the matrix P to be time-varying. This is because there is no guarantee that the matrix F has all its eigenvalues stable, which would mean that the conventional initial values P_1, which assumes that the system is in a stochastic steady state, cannot be obtained. Instead we make the assumption that the overall system is in stochastic steady state, and the time-varying Riccati equation is only relevant for the innovations process $\hat{\epsilon}_t$.

[12] In the simplest case, for two regression equations that depend on the same single shock, the covariance matrix of the shocks cannot, as is required, be inverted.

4.4 Nonfundamentalness of shocks under VAR estimation

A key issue in estimation is to be able to generate the theoretical responses to a fundamental shock. The implications of non-invertibility appear to have first been mentioned in the economics literature by Lippi and Reichlin (1994), and there has recently been renewed research activity, in response to the idea that although non-invertibility may be a serious problem for some shocks, it might not be for others.

The problem arises in the MA part of the VARMA representation of a DSGE model, and cannot occur if the model is represented by a stable VAR process. As we have seen earlier, the innovations process for an MA(1) process with a root outside the unit circle is identical to that of an MA(1) process with the inverse of that root. In the first case, the innovations process will (in the long term) be identical to the fundamental shock. However, if the root of the non-invertible MA(1) process is very close to the unit circle, then there will be little difference between the innovations process and the fundamental. Of course in such a case, the AR process that is estimated and is supposed to approximately replicate the MA(1) process will have to have many lags, or else there is a truncation problem (Soccorsi (2016)).

Much of the literature in this area (summarised by Alessi *et al.* (2011)) has concentrated on converting a non-fundamental representation from VAR estimation into a fundamental one by the use of Blaschke factors obtained from the underlying VARMA (or linearized DSGE) model that it is supposed to represent. More recently the focus has been on assessing whether non-invertibility can nevertheless allow for innovations to approximate fundamental shocks. Two methods are notable in this regard: Beaudry *et al.* (2016) recommend using the difference in variances between the innovations process and the fundamental shocks, motivated by (10) which in the perfect information case under rational expectations can be written as

$$\hat{\varepsilon}_t = m_t - Ez_{t,t-1} = E(z_t - z_{t,t-1}) = EA(z_{t-1} - z_{t-1.t-1}) + EB\varepsilon_t \tag{48}$$

Under invertibility, $z_{t-1} - z_{t-1.t-1}$ has a value of 0, so that regressing the innovations process $\hat{\varepsilon}_t$ on this latter term yields (in the scalar case) $R^2 = 0$. For the univariate case, in general we have $R^2 = 1 - var(\varepsilon_t)/var(\hat{\varepsilon}_t)$. In the multivariate case, $\text{cov}(\hat{\varepsilon}_t) =$

ESE', so that the departure of this from $\text{cov}(EB\varepsilon_t)$ yields a measure of how similar the innovations process is to the fundamental shocks. To address this on a shock-by-shock basis, one requires the Choleski decomposition of $ESE' = VV'$, or else a decomposition that depends for example on long run effects of each shock i.e. an SVAR decomposition. The corresponding R_i^2 for each shock is then given by

$$R_i^2 = 1 - u_{ii} \qquad U = V^{-1}EBB'E'(V')^{-1} = (u_{ij}) \tag{49}$$

The further is R_i^2 from 0, the worse is the fit.

4.4.1 A Multivariate Measure with Perfect Information

An obvious multivariate version of this is $R = I - V^{-1}EBB'E'(V')^{-1}$, and the maximum eigenvalue of R would then be a measure of the overall fit of the innovations to the fundamentals. In addition one can check whether any fundamentals can be perfectly identified by examining the eigenvalues of the difference between the variances of the innovations and and the fundamentals

$$\mathbb{B}^{PI} = ESE' - EBB'E' \tag{50}$$

Any zero eigenvalues coupled with the corresponding eigenvector will provide a means of decomposing the covariance matrix of the innovations ESE'.

Forni *et al.* (2016) suggest that one can use VARs as well for 'short systems', where the number of observables is smaller than the number of shocks. Utilising the underlying VARMA model, they suggest regressing the fundamental shocks against the innovations process, i.e., for the fundamental shock i, choose the least-squares vector m_i by minimizing the sum of squares of $\epsilon_{i,t} - m_i'\hat{\epsilon}_t$. Clearly, the theoretical value of this is

$$\hat{m}_i = cov(\hat{\epsilon}_t)^{-1}cov(\hat{\epsilon}_t, \epsilon_{i,t}) = (ESE')^{-1}(EB)_i \tag{51}$$

where $(EB)_i$ denotes the ith column of EB. A measure of goodness of fit is then

$$\mathbb{F}_i^{PI} = cov(\epsilon_{i,t}) - cov(\epsilon_{i,t}, \hat{\epsilon}_t)cov(\hat{\epsilon}_t)^{-1}cov(\hat{\epsilon}_t, \epsilon_{i,t}) = 1 - (EB)_i'(ESE')^{-1}(EB)_i \tag{52}$$

Thus one can as usual define a linear transformation of the $M\hat{\epsilon}_t$ (where M is made up of the rows m_i') as representing the fundamental shocks, but only take serious note of those shocks where the goodness of fit is close to 0. Once again, one can use the multivariate measure of goodness of fit

$$\mathbb{F}^{PI} = I - B'E'(ESE')^{-1}EB \tag{53}$$

where the diagonal terms then correspond to the terms $\mathbb{F}_i$ of (52). In (53) we note that $ESE' = cov(\hat{\epsilon}_t)$ from (23), and $(EB)_i = cov(\hat{\epsilon}_t, \epsilon_{i,t})$. The latter arises from subtracting (23) from (19) and noting that $\hat{\epsilon}_t = E(z_t - \bar{v}_t)$, and then subsequently deducing that $\mathbb{E}(z_{t+1} - \bar{v}_{t+1})\epsilon_{t+1}' = B$.

If the number of measurements is equal to the number of shocks, and if $\mathbb{F}_i = 0$ for all i, then since $\mathbb{F}^{PI}$ is by definition a positive definite matrix, it must be identically equal to 0. Of course, it may be the case that none of the $\mathbb{F}_i$ are zero, but that a linear combination of the fundamental shocks are exactly equal to a linear combination of the residuals. In addition, we might specify a particular value of the R^2 (e.g. $R_s^2 = 0.9$) fit of residuals to fundamentals such that we are happy to approximate the fundamental by the best fit of residuals.

Theorem 4. *Consider the more general case with the number of fundamental shocks possibly greater than the number of measurements. (a) All zero eigenvalues of* $\mathbb{F}^{PI}$ *correspond to a perfect fit between a linear combination of fundamentals and a best regression fit of residuals; (b) The number of eigenvalues of* $\mathbb{F}^{PI}$ *that are less than* $1 - R_s^2$ *correspond to the number of linear combinations of fundamentals that can be obtained approximately from the residuals.*

Proof. Both of these results follow from finding the best fit of a linear combination of fundamental shocks and residuals, which can be expressed as

$$min_{a,b}\mathbb{E}(a'\epsilon - b'\hat{\epsilon})^2 \;\; s.t. \;\; a'a = 1 \tag{54}$$

Given a, one obtains b via standard OLS techniques, and the problem reduces to minimizing $a'\mathbb{F}^{PI}a$ s.t. $a'a = 1$, with solution a equal to the eigenvector of the minimum eigenvalue of $\mathbb{F}^{PI}$. ■

The maximum eigenvalue of $\mathbb{F}^{PI}$ then provides a measure of overall non-fundamentalness.[13] It must of course be emphasised that none of these measures can be obtained directly from the data. The papers cited above all provide details of how simulations on the underlying VARMA models can indicate how to make appropriate inferences on the fundamental shocks using just the data and a VAR estimation.

4.4.2 A Multivariate Measure with Imperfect Information

Collard and Dellas (2004) and Collard and Dellas (2006) provide examples where there are large differences in the impulse response functions under imperfect and perfect information, and indeed Theorem 3 appears to indicate that this may be a major issue. In addition, Levine *et al.* (2012a), for an estimated DSGE model, find that such differences are quite large as well.

As we have seen for the perfect information case above, it is quite straightforward to obtain goodness of fit measures for the individual shocks from the multivariate measures, so for convenience we only list the latter. Firstly, the Beaudry *et al.* (2016) measure, which can be abbreviated to the difference between the variances of the innovations and the fundamentals, is given by

$$\mathbb{B}^{II} = EZE' - EBB'E' \tag{55}$$

Likewise, the multivariate Forni *et al.* (2016) measure can, after some effort, be written as

$$\mathbb{F}^{II} = I - B'D'(DPD')^{-1}DPE'(EZE')^{-1}EPD'(DPD')^{-1}DB \tag{56}$$

Analogously to the perfect information case, $EZE' = cov(\hat{\epsilon}_t)$, with $EPD'(DPD')^{-1}DB = cov(\hat{\epsilon}_t, \epsilon_t)$. The latter follows firstly because from (27) and (37) we can write $\hat{\epsilon}_t = E(z_{t,t-1} - \bar{v}_{1t}) + EPD'(DPD')^{-1}D\tilde{z}_t$. The first term is clearly independent of ϵ_t, while the covariance of the second term with ϵ_t is obtained by calculating $\mathbb{E}[\tilde{z}_{t+1}\epsilon'_{t+1}]$ in (25).

[13] This systems notion of fundamentalness is used elegantly by Canova and Sahneh (2017). They estimate a VAR, and test for non-fundamentalness by adding additional variables that depend on the variables of the VAR and also on other shocks that do not affect the VAR variables. Thus these additional variables depend on the fundamental shocks that affect the VAR variables. If the true VARMA model of the VAR model is not invertible, then the fundamental shocks can only be inferred as forward looking values of the residuals, so the test that they devise is whether the additional variables depend on these forward residuals.

In addition $\mathbb{F}_i^{II}$ corresponds to a measure of goodness of fit of the innovations residuals to the fundamental shocks, and provides information as to how well the VAR residuals correspond to the fundamentals. Note however that these measures correspond to the case when all observables are of current variables. While it is not straightforward to apply these ideas to the case when some variables are current and others are lagged, nevertheless we can apply them to the case when all variables are lagged. In particular, the theoretical value of $\mathbb{F}^{II,lagged}$ can now be defined as

$$\mathbb{F}^{II,lagged} = cov(\epsilon_t) - cov(\epsilon_t, \hat{\epsilon}_{t-1})cov(\hat{\epsilon}_{t-1})^{-1}cov(\hat{\epsilon}_{t-1}, \epsilon_t) \tag{57}$$

$cov(\hat{\epsilon}_{t-1})$ is of course equal to $cov(\hat{\epsilon}_t) = EZE'$, so the only change is to $cov(\hat{\epsilon}_{t-1}, \epsilon_t)$, which after a little effort can be derived as

$$\begin{aligned} cov(\hat{\epsilon}_{t-1}, \epsilon_t) &= EAPD(DPD)^{-1}DB - EAZE(EZE)^{-1}EPD(DPD)^{-1}DB \\ &\quad + EPD(DPD)^{-1}DFB - EPD(DPD)^{-1}DFPD(DPD)^{-1}DB \end{aligned} \tag{58}$$

Then the fit $\mathbb{F}_i^{II,lagged}$ to the ith shock is just given by the ith main diagonal term of $\mathbb{F}^{II,lagged}$.

In a later section we compare numerically these perfect and imperfect information multivariate measures of the fit of the innovations to the fundamentals for a DSGE model.

4.4.3 The Fundamentalness of Single Shocks

Up to now our results are for invertibility of the vector of fundamental shocks ε_t. However in the empirical literature using VARs it is common to focus on just one shock such as in the examination of the hours-technology question in Gali (1999). Beaudry *et al.* (2016), Forni *et al.* (2016) and Canova and Sahneh (2017) show that one can recover a fundamental shock with a reduced-form VAR even when the system as a whole (as measured by our multi-variate measures) is non-fundamental.

Canova and Sahneh (2017) have recently produced one of the best empirical tests of non-fundamentalness of VARs. It is based on the insight that if we observe an MA(1) process

$$y_t = \epsilon_t + \theta\epsilon_{t-1} \qquad \text{where } \theta > 1 \tag{59}$$

then this will be estimated as $y_t = e_t + \theta^* e_{t-1}$, where the consistent parameter estimate θ^* satisfies $\theta^* = 1/\theta$ as $t \to \infty$.

It follows that ϵ_t cannot be recovered from past values of e_t because the expansion of $(1+\theta L)^{-1}$ in powers of L is an unbounded infinite sum. However by writing y_t as

$$y_t = (1+\theta^* L)e_t = (1+\theta L)\epsilon_t = \theta L(1+\theta^* L^{-1})\epsilon_t \tag{60}$$

it follows that ϵ_t can be retrieved as a bounded infinite sum of future values of e_t, or as a bounded infinite sum of future values of y_t.

Canova and Sahneh (2017) utilise this well-known result by introducing a new observed variable. Thus, suppose that the underlying variables of interest are contained in the vector x_t which is a VARMA in the shocks v_t, and that the VARMA is approximated by a VAR, which is estimated in the form $A(L)x_t = \hat{v}_t$. Now introduce a new variable $z_t = B(L)v_t + C(L)w_t$, which is dependent on the same shocks v_t as the main variables x_t, and some additional shocks w_t. If there is no invertibility problem for x_t estimated as a VAR, then z_t can be rewritten (as $t \to \infty$) as

$$z_t = B(L)\hat{v}_t + C(L)w_t \tag{61}$$

If there is an invertibility problem then at least one element of v_t depends on future values of $\hat{v}_t$. Thus conducting a standard Granger causality test of whether z_t depends on future values of the recorded residuals $\hat{v}_t$ is sufficient to deduce whether the latter are fundamental or not. One can of course use this test one-by-one on each element of $\hat{v}_t$, so as to determine which estimated VAR shock is the non-fundamental one.

Thus the above procedure, which determines fundamentalness from an empirical perspective, complements ours, which determines fundamentalness from the belief in an underlying model.

5 Imperfect Information and Aggregation

In this section we briefly examine the relevance of the assumptions on imperfect information that we have investigated so far, and how they can relate to a situation where there are a number of agents with idiosyncratic information.

In models with imperfect information it is important to distinguish between *idiosyncratic* uncertainty (at the level of the household or firm) and *aggregate* uncertainty. Our solution procedure above applies to the aggregate model but what happens to idiosyncratic uncertainty?

Consider the first order conditions for firm i who uses a CD technology

$$Y_{i,t} = (A_{i,t}A_tH_{i,t})^{\alpha}K_{i,t-1}^{1-\alpha} \tag{62}$$

where A_t is an aggregate technology process and $A_{i,t}$ is a firm-specific idiosyncratic process. Since $H_{i,t}$ and $K_{i,t-1}$ are decision variables, it follows that each firm must know its own composite labour productivity $A_{i,t}A_t$. If there is no idiosyncratic uncertainty then in our representative agent model, A_t, the aggregate technology process, will be observed by firms and households who own them. Then with n firms, in aggregate $Y = nY_{i,t}$, $K = nK_{i,t}$ and $H_t = nH_{i,t}$ (total demand for labour). But since we have constant returns to scale our aggregate production function becomes

$$Y_t = (A_tH_t)^{\alpha}K_{t-1}^{1-\alpha} \tag{63}$$

as before, but here we must stress that the aggregate technology process *must be observed*.

Our II solution procedure for both solution and estimation only hold up to first order. Thus log-linearizing (62) we have

$$y_{i,t} = \alpha(a_{i,t} + a_t + h_{i,t})) + (1-\alpha)k_{i,t-1} \tag{64}$$

where $x_t \equiv \log(X_t/X) \approx \frac{X_t - X}{X}$.

Assume an AR1 process for a_t:

$$a_{t+1} = \rho_a a_t + \epsilon_{a,t+1} \tag{65}$$

$$a_{i,t} = \epsilon_{a_i,t} \tag{66}$$

where $\epsilon_{a,t} \sim n.i.i.d(0, \sigma_{\epsilon^2_{a,t}})$ and $\epsilon_{a_i,t} \sim n.i.i.d(0, \sigma_{\epsilon^2_{a_i,t}})$. Exploiting ergodicity we have

$E_i[a_{i,t}] = \mathbb{E}_t[a_{i,t}] = 0$. Then aggregating we arrive at

$$y_t = \alpha(a_t + h_t) + (1 - \alpha)k_{t-1} \tag{67}$$

where $y_t \equiv E_i[y_{i,t}]$ etc. Thus *idiosyncratic uncertainty disappears in the aggregate model* up to first order.

However firm i can form expectations of the aggregate technology process using observations of $a_t^{obs} = \frac{y_{i,t}-(1-\alpha)k_{i,t-1}}{\alpha} = a_t + a_{i,t}$. This is given by the Kalman Filter

$$\begin{aligned} \mathbb{E}_t[a_t] &= E_{t-1}[a_t] + J_a(a_t^{obs} - \mathbb{E}_{t-1}[a_t]) && (68) \\ &= (1 - J_a)\rho_a a_{t-1} + J_a a_t^{obs} && (69) \end{aligned}$$

where the Kalman gain in this case is simply

$$J_a = \frac{\sigma_{\epsilon_a}^2}{\sigma_{\epsilon_a}^2 + \sigma_{\epsilon_{a_i}}^2} \tag{70}$$

In the limit as the signal for a_t becomes very noisy (i.e., $\frac{\sigma_{\epsilon_{a_i}}}{\sigma_{\epsilon_a}} \to \infty$), $J_a \to 0$ and the firm-specific addition to the information set adds nothing. This is the limiting case considered in Pearlman *et al.* (1986), Levine *et al.* (2012a) and here, and is a special case of Nimark (2008) where the variance of idiosyncratic shock component greatly exceeds that of the aggregate component so the former yields no information about the latter.

Our limiting case contrasts with Ilut and Saijo (2016) that uses a heterogeneous agent model where idiosyncratic uncertainty does not vanish at the aggregate level as in our framwork. Whereas our focus is on uncertainty and imperfect information at the aggregate levels, these authors focus on the idiosyncratic level. These two frameworks are complementary and both in fact generate persistence, hump-shaped dynamics and positive co-movement of consumption and output to demand shocks.

6 Applications: Imperfect Information in the RBC and NK Models

This section illustrates our theoretical results using RBC and NK model. We first consider the standard RBC model with a non-zero growth steady state. Then we add a nominal side to investigate the possibility that II allows for real effects of monetary policy.

6.1 The Baseline Model

For the household:

$$\text{Utility} : U_t = U(C_t, L_t) \tag{71}$$

$$\text{Euler Consumption} : U_{C,t} = \beta R_t \mathbb{E}_t \left[U_{C,t+1} \right] \tag{72}$$

$$\text{Labour Supply} : \frac{U_{H,t}}{U_{C,t}} = -\frac{U_{L,t}}{U_{C,t}} = -W_t \tag{73}$$

$$\text{Leisure and Hours} : L_t \equiv 1 - H_t \tag{74}$$

where C_t is real consumption, L_t is leisure, R_t is the gross real interest rate set in period t to pay out interest in period $t+1$, H_t are hours worked and W_t is the real wage.

The Euler consumption equation, (72), where $U_{C,t} \equiv \frac{\partial U_t}{\partial C_t}$ is the marginal utility of consumption and $\mathbb{E}_t[\cdot]$ denotes rational expectations based on the agents' information set, describes the optimal consumption-savings decisions of the household. It equates the marginal utility from consuming one unit of income in period t with the discounted marginal utility from consuming the gross income acquired, R_t, by saving the income. For later use define $\Lambda_{t,t+1} \equiv \beta \frac{U_{C,t+1}}{U_{C,t}}$ is the *real stochastic discount factor* over the interval $[t, t+1]$. (73) equates the real wage with the marginal rate of substitution between consumption and leisure.

Output and the firm behaviour is summarized by:

$$\text{Output} : Y_t = F(A_t, H_t, K_{t-1}) \tag{75}$$

$$\text{Labour Demand} : F_{H,t} = W_t \tag{76}$$

$$\text{Capital Demand} : 0 = \mathbb{E}_t \left[\Lambda_{t+1} (F_{K,t+1} - R_t + 1 - \delta) \right] \tag{77}$$

$$\text{Stochastic Discount Factor} : \Lambda_t = \beta \frac{U_{C,t+1}}{U_{C,t}} \tag{78}$$

(75) is a production function where K_t is *end-of-period* t capital stock. Equation (76), where $F_{H,t} \equiv \frac{\partial F_t}{\partial H_t}$, equates the marginal product of labour with the real wage. (77), where $F_{K,t} \equiv \frac{\partial F_t}{\partial K_{t-1}}$, equates the marginal product of capital with the cost of capital. The model is completed with an output equilibrium, law of motion for capital and a balanced budget constraint with fixed lump-sum taxes.

$$Y_t = C_t + G_t + I_t \tag{79}$$

$$I_t = K_t - (1-\delta)K_{t-1} \tag{80}$$

$$G_t = T_t \tag{81}$$

We now specify functional forms for production and utility and AR(1) processes for exogenous variables A_t and G_t. For production we assume a Cobb-Douglas function. The consumers' utility function is non-separable and consistent with a balanced growth path when the inter-temporal elasticity of substitution, $1/\sigma_c$ is not unitary. These functional forms, the associated marginal utilities and marginal products, and exogenous processes are given by

$$F(A_t, H_t, K_{t-1}) = (A_t H_t)^{\alpha} K_{t-1}^{1-\alpha} \tag{82}$$

$$F_H(A_t, H_t, K_{t-1}) = \frac{\alpha Y_t^W}{H_t} \tag{83}$$

$$F_K(A_t, H_t, K_{t-1}) = \frac{(1-\alpha) Y_t^W}{K_{t-1}} \tag{84}$$

$$\log A_t - \log \bar{A}_t = \rho_A (\log A_{t-1} - \log \bar{A}_{t-1}) + \epsilon_{A,t} \tag{85}$$

$$\log G_t - \log \bar{G}_t = \rho_G (\log G_{t-1} - \log \bar{G}_{t-1}) + \epsilon_{G,t} \tag{86}$$

$$U_t = \frac{(C_t^{(1-\varrho)} L_t^{\varrho})^{1-\sigma_c} - 1}{1-\sigma_c} \tag{87}$$

$$U_{C,t} = (1-\varrho) C_t^{(1-\varrho)(1-\sigma_c)-1} (1-H_t)^{\varrho(1-\sigma_c)} \tag{88}$$

$$U_{H,t} = -\varrho C_t^{(1-\varrho)(1-\sigma_c)} (1-H_t)^{\varrho(1-\sigma_c)-1} \tag{89}$$

(71) – (89) describe an equilibrium in U_t, C_t, W_t, Y_t, Y_t^W, L_t, H_t, K_t, I_t, R_t, T_t, given A_t and G_t where for the latter we assume AR(1) processes about steady states $\bar{A}, \bar{G}$ driven

by zero mean iid shocks $\epsilon_{A,t}$ and $\epsilon_{G,t}$.

6.2 Simulations with Imperfect Information

The purpose of these simulations is to show how imperfect information acts as a 'parsimonious friction' generating persistent, hump-shaped dynamics and positive co-movement to demand shocks without relying on the usual persistence mechanisms such as habit and investment adjustment costs. Indeed in the RBC model with monetary policy we show how II can generate a real effect of a monetary shock.

We first consider the RBC model without a nominal side and without habit and investment adjustment costs, **Model 1**. With two shock processes, A_t and G_t, the following combinations of two observables (from a set of observables: $(Y_t, C_t, I_t, H_t, W_t, R_t)$) result in no difference between perfect and imperfect solution procedures: (Y_t, C_t), (Y_t, I_t), (Y_t, W_t) and (Y_t, H_t). On the other hand, the following combinations do generate a difference: (Y_t, R_t), (W_t, R_t) and (C_t, R_t).

Example 2. Table 1 below summarises a complete set of combinations for this model, i.e., $c = \frac{6!}{(6-2)!6!} = 15$, based on the various full rank and stability conditions of Theorem 2 and the rank condition of Corollary 1 stated in Section 4.2 (a necessary and sufficient condition for invertibility).

Combinations of observables (where EB is of full rank)	Theorem 2 & Corollary 1	Description
$(Y_t, C_t), (Y_t, I_t), (Y_t, H_t), (Y_t, W_t)$ $(C_t, I_t), (C_t, H_t), (C_t, W_t)$ $(I_t, H_t), (I_t, W_t), (H_t, W_t)$	D is of full (row) rank DB is of full (row) rank	System is invertible and II equivalent to perfect info.
$(Y_t, R_t), (C_t, R_t), (I_t, R_t)$ $(W_t, R_t), (H_t, R_t)$	D is rank deficient DB is rank deficient	System is not invertible and II not equivalent to perfect info.
Lagged observations	D is of full (row) rank DB is rank deficient	System is not invertible and II not equivalent to perfect info.

Table 1: **Summary of Invertibility Condition for Model 1**

Figures 2 and 3 show the deterministic IRFs in response to unanticipated shocks A_t and G_t in this case with no built-in persistence mechanisms. If observations are made with a lag this always leads to a failure of the rank condition as shown in Table 1. Figures 4–5 show an example of IRFs for such a case. In both sets of IRFs we see that II induces endogenous persistence, hump-shaped persistence and co-movement of consumption and demand. This

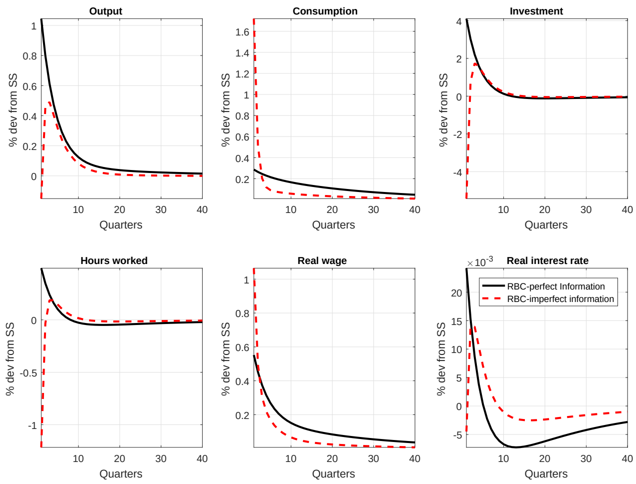

Figure 2: Model 1 Impulse Responses to a Technology Shock, A_t. Observables Y_t, R_t

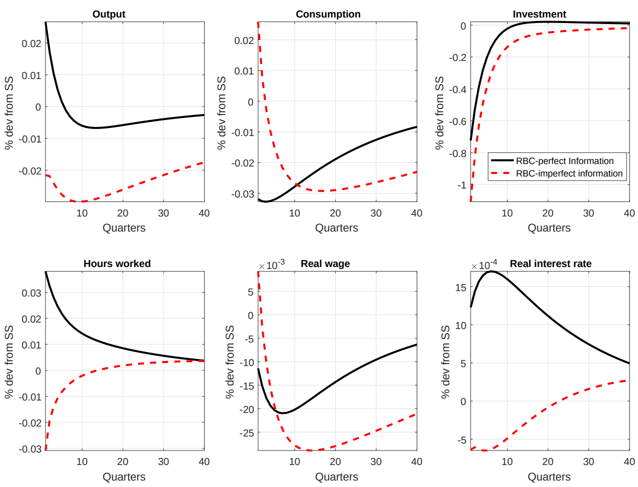

Figure 3: Model 1 Impulse Responses to a Government Spending Shock, G_t. Observables Y_t, R_t

is essentially due to a signal processing problem, that agents cannot immediately tell from their measurements which is the exact source of the shock. Technically, this is when the full rank condition, which explains the effect of current shocks, fails to hold. Even when one observes these measurements over infinite time, one cannot recover the exact values of the shocks.

Secondly in this section we examine the possibility of II generating real effects of

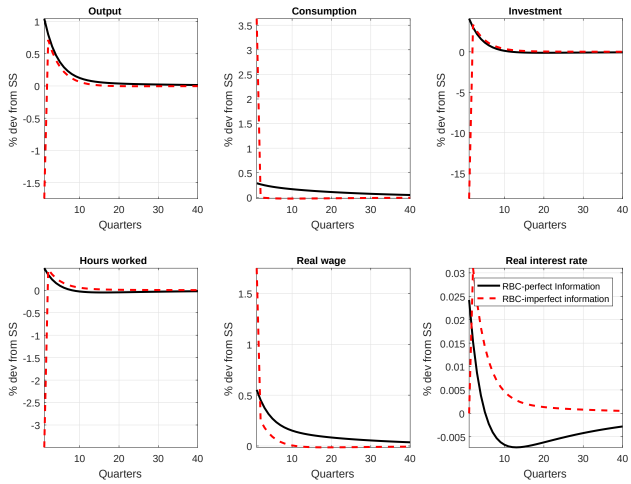

Figure 4: Model 1 Impulse Responses to a Technology Shock, A_t. Observables Y_{t-1}, C_{t-1}

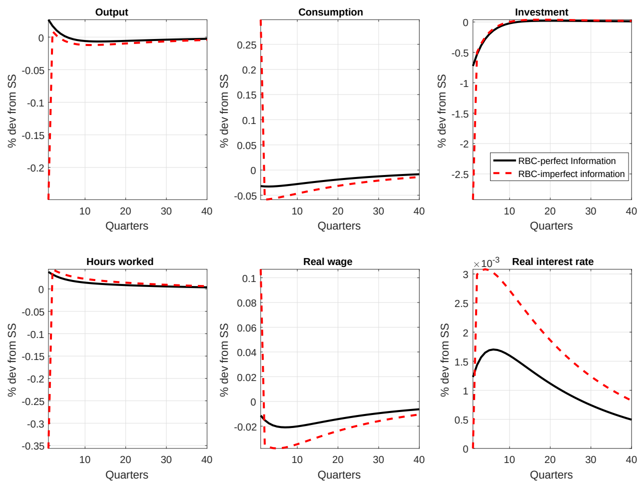

Figure 5: Model 1 Impulse Responses to a Government Spending Shock, G_t. Observables Y_{t-1}, C_{t-1}

monetary policy in a RBC model. To pursue this we augment the model with a Fischer equation

$$R_t = \frac{R_{n,t-1}}{\Pi_t} \tag{90}$$

where R_t is now the ex post gross real interest rate with $\mathbb{E}_t R_{t+1}$ the ex ante expected real interest rate. Bonds are no longer riskless as we now have inflation risk and the Euler equation now becomes

$$1 = \mathbb{E}_t\left[\Lambda_{t,t+1} R_{t+1}\right] \tag{91}$$

The nominal interest rate is given by the following Taylor-type rule

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\pi \log\left(\frac{\Pi_t}{\Pi_{targ,t}}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right) + \epsilon_{M,t} \tag{92}$$

with the inflation target $\Pi_{targ,t}$ following an AR(1) process and a white noise monetary policy shock $\epsilon_{M,t}$. This makes the implicit assumption that the central bank observes Π_t and Y_t along with the inflation target. If the central bank has the same information regarding inflation and output as the economic agents (92) is replaced with

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\pi \log\left(\frac{\Pi_{t,t}}{\Pi_{targ,t}}\right) + \theta_y \log\left(\frac{Y_{t,t}}{Y}\right)\right) + \epsilon_{M,t} \tag{93}$$

In addition, we add the Smets and Wouters (2007) form of investment adjustment costs to the RBC model. The law of motion for capital becomes

$$\begin{aligned} K_t &= (1-\delta)K_{t-1} + (1-S(X_t))I_t\,;\;\; S',\, S'' \geq 0\,;\;\; S(1) = S'(1) = 0 \\ X_t &\equiv \frac{I_t}{I_{t-1}} \end{aligned}$$

We introduce capital producing firms that at time t convert I_t of output into $(1-S(X_t))I_t$ of new capital sold at a real price Q_t and then maximize with respect to $\{I_t\}$ expected discounted profits. The first-order condition for the capital producers is

$$Q_t(1 - S(X_t) - X_t S'(X_t)) + E_t\left[\Lambda_{t,t+1}\, Q_{t+1} S'(X_{t+1}) X_{t+1}^2\right] = 1$$

Demand for capital by the wholesale firm owned by households is now given by

$$\begin{aligned} 1 &= R_t \mathbb{E}_t[\Lambda_{t,t+1}] \\ &= \frac{\mathbb{E}_t\left[\Lambda_{t,t+1}[(1-\alpha)\frac{Y_{t+1}^W}{K_t} + (1-\delta)Q_{t+1}]\right]}{Q_t} \equiv \mathbb{E}_t[\Lambda_{t,t+1} R_{K,t+1}] \end{aligned} \tag{94}$$

In (94) the right-hand-side is the discounted gross return to holding a unit of capital in from t to $t+1$. The left-hand-side is the discounted gross return from holding bonds, the opportunity cost of capital. Note that without investment costs, $S = 0$, $Q_t = 1$ (94) reduces to the standard Euler equation. We complete this set-up with the functional form for investment adjustment, $S(X) = \phi_X(X_t - 1)^2$, which completes the RBC model augmented with capital producers and monetary policy.

We now solve and simulate this model (**Model 2**) again under imperfect information. We calibrate the parameters consistent with an observed baseline steady state equilibrium (e.g. factor shares, hours worked etc.). Some parameter values are set roughly reflecting the empirical literature (e.g. the adjustment costs in Smets and Wouters (2007)). We first show that using an example of a standard RE model when the invertibility condition fails under II this requires the iterative reduction algorithm when converting the II state space to the Blanchard-Kahn form described in Appendix B (and Theorem 2(b)). This procedure is required to yield a suitable reduced-form system which is to be processed via the Kalman filter to obtain the likelihood function for estimation purposes.

Example 3. For this model example the algorithm set out in Appendix B reports that A_0 in the original set-up is singular so that one has to re-define forward-looking variables that are linear combinations of the original Y_t, to yield the smallest number of equations that involve forward-looking expectations. In particular, the following algorithm is implemented:

1. Obtain the singular value decomposition for matrix A_0 and partitions of A_0 from (A.3)
$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t$$

2. Transform (A.3) to forward-looking subsystem and and re-define forward-looking system matrices F_i, $i = 1, ..., 5$ according to Stage 2 in Appendix B

3. Transform (A.3) to backward-looking subsystem and and re-define backward-looking system matrices C_i, $i = 1, ..., 5$ according to Stage 3 in Appendix B

4. At this stage, the calibrated model reports that $C_2 + G_2$ is not invertible, where G_2 is the matrix associated with $s_{t,t}$ (s_t defines the backward-looking states in the system), Stage 4 in Appendix B is now required to be iterated once to reduce the dimension of the forward-looking F matrices by 1 and increase the dimension of the

backward-looking C matrices by the same amount (this is done through the reduction procedure (A.13)-(A.16)). Re-define C_2, $C_2 + G_2$ using (A.16) which is now of full rank. As a general case, some models may require this stage to be repeated up to a finite number of times until $C_2 + G_2$ is non-singular

5. Generate C_2^{-1} and $(C_2 + G_2)^{-1}$, proceed to the following stages, and we have the required Blanchard-Kahn form set out by (A.19) and (A.20).

Finally, Figures 6–7 show the IRFs to the two additional policy shocks, $\epsilon_{M,t}$ and $\Pi_{targ,t}$, with three observables $Y_t, R_{n,t}, \Pi_t$ observed under II. We assume that the nominal interest rate set by the central bank is observed immediately and consider the following two observation sets: $(Y_t, R_{n,t}, \Pi_t)$ and $(Y_{t-1}, R_{n,t}, \Pi_{t-1})$. In the latter case, it takes time for agents to observe both aggregate output and inflation. The IRFs show II *can* generate real effects albeit on a small scale. Monetary policy can produce real effects on output and employment in this RBC economy if there is a 'rigid' learning process from the agents who realise gradually about the shocks and states of the system through Kalman updating. Since the II agents observe economic conditions with delayed information, the market does not adjust instantly to the monetary shocks (the demand shifts more broadly) and as noted this generates endogenous inertia in the system. Rationalising rigidity in this way is more direct, parsimonious, and a less difficult theoretical problem without heavily relying on microeconomic foundations.

6.3 Quantitative Measures of Nonfundamentalness

We now consider and implement the multivariate measure of goodness of fit set out in Section 4.4. We compare numerically the perfect and imperfect information measures of the fit of the innovations to the fundamentals for Model 1 (the baseline). Our focus is on (53) and (56), the corresponding measures of correlation between $\hat{\epsilon}_t$ and ϵ_t, for the perfect and imperfect information cases, respectively where $cov(\hat{\epsilon}_t) = ESE'$ and $cov(\hat{\epsilon}_t) = EZE'$ are the covariance matrices of the innovation processes for the two cases. $cov(\epsilon_t)$ is for the fundamental shocks in the model. As noted, the maximum eigenvalue provides a measure of overall non-fundamentalness. In addition, any zero eigenvalues provide information as to which fundamental shocks can be satisfactorily identified (i.e. evidence of partial sufficiency of individual shocks in the system). The following example checks the difference

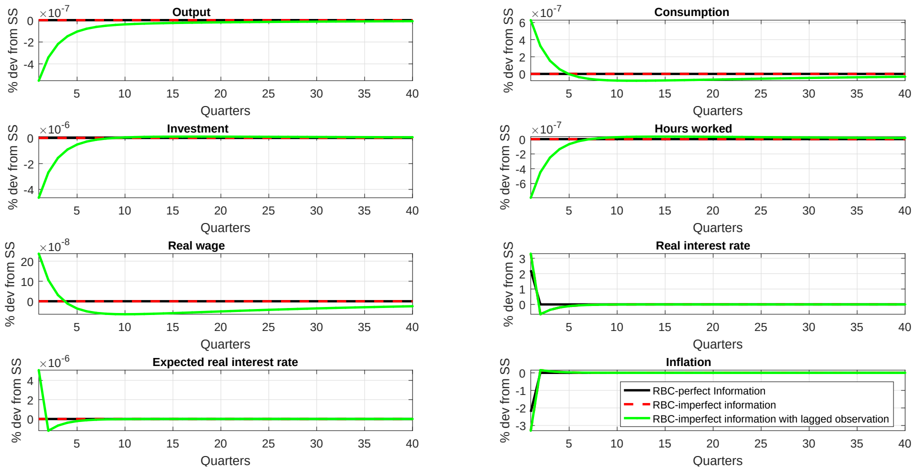

Figure 6: Monetary RBC Model 2. Impulse Responses to a Monetary Policy Shock, $\epsilon_{M,t}$. Observables $Y_t, R_{n,t}, \Pi_t$ and $Y_{t-1}, R_{n,t}, \Pi_{t-1}$

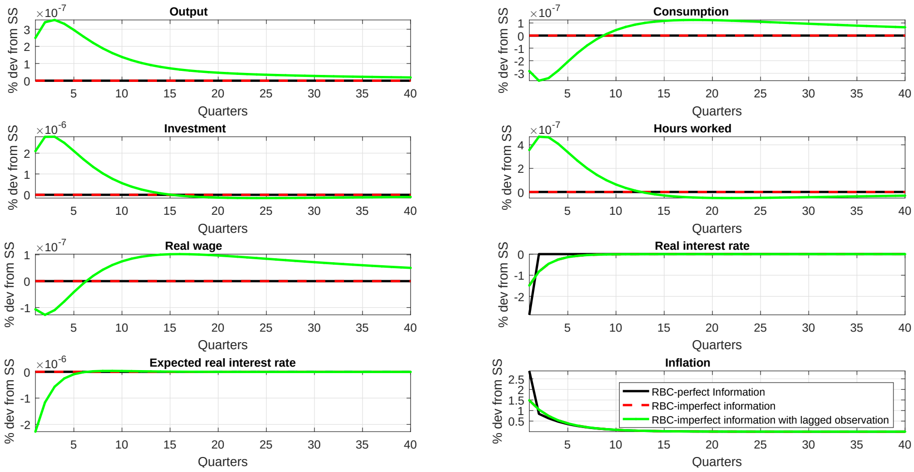

Figure 7: Monetary RBC Model 2. Impulse Responses to an Inflation Target Shock, $\Pi_{targ,t}$. Observables $Y_t, R_{n,t}, \Pi_t$ and $Y_{t-1}, R_{n,t}, \Pi_{t-1}$

between perfect and imperfect information in terms of identifying the fundamentals from the perspective of VARs via the eigenvalues of R^{PI} and R^{II}, assuming that Model 1 is the DGP.

Example 4. For convenience, we assume that the shocks A_t and G_t are normalized such that $cov(\epsilon_t) = I$. The parameter calibration remains the same as in the previous simulation. Table 2 below summarises the results from the same set of observable combinations for this model as in Table 1.

For the case of the system being invertible, and EB is of full rank, the solutions of (24)

Combinations of observables (where EB is of full rank)	Theorem 2 & Corollary 1	Eigenvalues	Diagonal values of $\mathbb{F}$
$(Y_t, C_t), (Y_t, I_t), (Y_t, H_t), (Y_t, W_t)$ $(C_t, I_t), (C_t, H_t), (C_t, W_t)$ $(I_t, H_t), (I_t, W_t), (H_t, W_t)$	D,DB are of full row rank (=2) AND $A(I - B(EB)^{-1}E)$ is stable rank(D)=rank(E)=2	$\mathbb{F}^{PI} = 0$ $eig(\mathbb{F}^{PI}) = (0, 0)$	
II not equivalent to perfect info.			
(Y_t, R_t)	D,DB are rank deficient (=1)	$\mathbb{F}^{PI} = 0$, $eig(\mathbb{F}^{II}) = (0, 1)$	$\mathbb{F}_i^{PI} = [0, 0]$, $\mathbb{F}_i^{II} = [0.0001, 0.9999]$
(C_t, R_t)	D,DB are rank deficient (=1)	$\mathbb{F}^{PI} = 0$, $eig(\mathbb{F}^{II}) = (0.0027, 1)$	$\mathbb{F}_i^{PI} = [0, 0]$, $\mathbb{F}_i^{II} = [0.0028, 0.9999]$
(I_t, R_t)	D,DB are rank deficient (=1)	$\mathbb{F}^{PI} = 0$, $eig(\mathbb{F}^{II}) = (0.0691, 1)$	$\mathbb{F}_i^{PI} = [0, 0]$, $\mathbb{F}_i^{II} = [0.2727, 0.7964]$
(W_t, R_t)	D,DB are rank deficient (=1)	$\mathbb{F}^{PI} = 0$, $eig(\mathbb{F}^{II}) = (0.0010, 1)$	$\mathbb{F}_i^{PI} = [0, 0]$, $\mathbb{F}_i^{II} = [0.0010, 1]$
(H_t, R_t)	D,DB are rank deficient (=1)	$\mathbb{F}^{PI} > 0$, $eig(\mathbb{F}^{II}) = (0.8906, 1)$	$\mathbb{F}_i^{PI} = [0.0057, 0.6597]$, $\mathbb{F}_i^{II} = [0.8929, 0.9976]$
One observation: rank(D)=rank(E)=1			
(C_t)	DB is not of full row rank (=1)	$eig(\mathbb{F}^{II}) = (0.9712, 1)$	$\mathbb{F}_i^{PI} = [0.0126, 0.9877]$, $\mathbb{F}_i^{II} = [0.0172, 0.9999]$
(I_t)	DB is not of full row rank (=1)	$eig(\mathbb{F}^{II}) = (0.4376, 1)$	$\mathbb{F}_i^{PI} = [0.0301, 0.9699]$, $\mathbb{F}_i^{II} = [0.4772, 0.8536]$
(H_t)	DB is not of full row rank (=1)	$eig(\mathbb{F}^{II}) = (0.8689, 1)$	$\mathbb{F}_i^{PI} = [0.0206, 0.9942]$, $\mathbb{F}_i^{II} = [0.9930, 0.9998]$
(W_t)	DB is not of full row rank (=1)	$eig(\mathbb{F}^{II}) = (0.9890, 1)$	$\mathbb{F}_i^{PI} = [0.0010, 0.9996]$, $\mathbb{F}_i^{II} = [0.0065, 1]$
(Y_t)	DB is not of full row rank (=1)	$eig(\mathbb{F}^{II}) = (0, 1)$	$\mathbb{F}_i^{PI} = [0.0007, 0.9993]$, $\mathbb{F}_i^{II} = [0.0001, 0.9999]$
Lagged observations: rank(D)=rank(E)=2			
(Y_{t-1}, C_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9776, 0.8908)$	$\mathbb{F}_i^{II} = [0.9776, 0.8908]$
(Y_{t-1}, I_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9776, 0.0474)$	$\mathbb{F}_i^{II} = [0.9776, 0.0475]$
(Y_{t-1}, H_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9776, 0.8908)$	$\mathbb{F}_i^{II} = [0.9776, 0.8908]$
(Y_{t-1}, W_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9776, 0.8908)$	$\mathbb{F}_i^{II} = [0.9776, 0.8908]$
One lagged observation: rank(D)=rank(E)=1			
(C_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9771, 1)$	$\mathbb{F}_i^{II} = [0.9771, 1]$
(I_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9768, 1)$	$\mathbb{F}_i^{II} = [0.9819, 0.9949]$
(H_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9967, 1)$	$\mathbb{F}_i^{II} = [0.9968, 0.9999]$
(W_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9773, 1)$	$\mathbb{F}_i^{II} = [0.9773, 1]$
(Y_{t-1})	DB is not of full row rank (=0)	$eig(\mathbb{F}^{II}) = (0.9776, 1)$	$\mathbb{F}_i^{II} = [0.9776, 1]$

Table 2: **Summary of Nonfundamentalness Tests for Model 1.** Order of shocks: A_t, G_t

and (28) are $S = P = BB'$ and, from which it follows that $\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$, and the two processes are perfectly correlated across the PI and II cases. This shows that invertibility or fundamentalness can allow for innovations to exactly approximate fundamental shocks. For the case of non-invertibility, the further is $\mathbb{F}^{II}$ from 0, the worse is the fit. Examples $(Y_t, R_t), (C_t, R_t), (I_t, R_t), (W_t, R_t)$ in the table show the cases while the perfect information solution is invertible (or there is complete fundamentalness, i.e. $\mathbb{F}^{PI} = 0$) the imperfect information counterparts are not (i.e. $\mathbb{F}^{II} > 0$ in the positive definite sense). With the same observables, solving the reduced form system (under perfect information) through (23), the solution of (24) gives $S - BB' = 0$, from which this means $\mathbb{F}^{PI} = 0$. Solving the steady state Riccati (28) for our case of imperfect information, we have $S = P > BB'$ and it automatically follows that $\mathbb{F}^{II} > \mathbb{F}^{PI}$. Therefore, interestingly we show that the simple RBC model introduces non-fundamentalism with the same measurements under II, rather than PI.

The only way to decide the overall fit of the RBC model approximating the fundamentals by the innovations process is to determine the maximum eigenvalue of $\mathbb{F}^{II}$. In Table 2, the II fit of the innovations to the structural shocks is very poor as the eigenvalues are all far from 0, when DB is not of full row rank, except for the case when the symmetric limited information set is contemporaneous, in which case, the first eigenvalue being very close to

0 (e.g. with $((Y_t, R_t))$, it is $2.79e-06$) indicates partial fundamentalness or that one of the the two shocks may be satisfactorily identified in this model. Assuming that a simple baseline RBC model is the DGP from which potentially VARs and SVARs are identified, this diagnostic result remarkably and strongly underlines our Theorem 3. When there are large differences in the impulse response functions under imperfect and perfect information, non-fundamentalness may be quantitatively severe, indeed according to Theorem 3, the simulation appears to indicate that this may be a major issue. However, if agents and the econometrician are assumed to observe (Y_{t-1}, I_{t-1}) in their model, the eigenvalue approach clearly indicates that the fit for non-fundamentalness is significantly improved ($eig(\mathbb{F}^{II}) = (0.0474, 0.9776)$) for the system of non-invertibility, suggesting an much improved quantitative importance in evaluating its cyclical relevance from the underlying DGP and in assessing how valid the VAR is for identifying this particular structural shock when agents have limited information. Finally, as expected, the overall fit also depends on the ratio of observables to shocks, in other words, the fewer the observations made by the agents compared to shocks (one observable in Table 2), the less well do VARs perform.

The last column of Table 2 reports the diagonal values of the (non-zero) $\mathbb{F}^{PI}$ and $\mathbb{F}^{II}$ matrices. These tell us explicitly about the goodness of fit of the residuals to the fundamental shocks. Any zero values reported in the diagonal matrices indicate an exact fit for the corresponding individual shocks in the models. This exercise is also extended to the following Smets and Wouters (2007) models. Clearly, the goodness of fit deteriorates when switching from PI to II, and more significantly depending on the size of the model and the number of shocks included.

But what about the cases when we may have non-invertibility under perfect information? Testing for non-fundamentalness for a number of structural shocks can be achieved by looking at more complicated larger-sized models.

Example 5. We run our simulation exercise using a version of Smets and Wouters (2007) model (henceforth SW). This model is selected because it features a number of nominal and real frictions in order to closely mimic the pattern of real aggregate variables, inflation and interest rate. There are seven structural shocks in SW. The model has five AR(1) processes, for the shocks on government spending, technology, preference, investment specific, monetary policy, and two ARMA(1,1) processes, for price and wage markup. In this exercise, we skip the description of the model and slightly modify the model by gradually adding more shocks. The SW model is estimated based on seven quarterly macroeconomic time series. When we assume that this exactly coincides with the agents' limited information set so in effect the number of measurements is equal to the number of shocks and EB is non-singular (Case 1: Original SW). In the modified versions of the model, the only changes we make are that (1) we add an inflation target shock so the number of shocks exceeds the number of observables (Case 2: SW with 8 shocks); (2) we further add measurement errors to the observations of real variables and inflation (Case 3: SW with 13 shocks). Table 3 summarises the key results from the simulation, based on Theorems 2 and the test for nonfundamentalness.

	Case 1: Original SW	Case 2: SW with Inflation Obj.		Case 3: SW with MEs	
	Measurements = Shocks (=7)	8 Shocks		13 Shocks	
Theorem 2 Corollary 1	D, DB full row rank $A(I - B(EB)^{-1}E)$ stable	EB not invertible		EB not invertible	
Goodness of Fit	$\mathbb{F}^{PI} = \mathbb{F}^{II} = 0$	$\mathbb{F}^{PI}_{(8\times 8)}$	$\mathbb{F}^{II}_{(8\times 8)}$	$\mathbb{F}^{PI}_{(13\times 13)}$	$\mathbb{F}^{II}_{(13\times 13)}$
Eigenvalues	$eig(\mathbb{F}^{PI}) = eig(\mathbb{F}^{II}) = 0$	$\begin{bmatrix} 1.0000 \\ 0.0013 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1.0000 \\ 0.0016 \\ 0.0009 \\ 0.0001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.0971 \\ 0.0454 \\ 0.0138 \\ 0.0001 \\ 0.0019 \\ 0.0058 \\ 0.0100 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5404 \\ 0.8182 \\ 0.3627 \\ 0.2975 \\ 0.0302 \\ 0.0011 \\ 0.0044 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
Diagonal values		$\begin{bmatrix} -0.0000 \\ 0.0006 \\ 0.0000 \\ 0.0005 \\ 0.0245 \\ 0.0000 \\ 0.0001 \\ 0.9756 \end{bmatrix}$	$\begin{bmatrix} 0.0000 \\ 0.0006 \\ 0.0000 \\ 0.0004 \\ 0.0256 \\ 0.0000 \\ 0.0001 \\ 0.9761 \end{bmatrix}$	$\begin{bmatrix} 0.2216 \\ 0.0924 \\ 0.5199 \\ 0.1600 \\ 0.1007 \\ 0.2262 \\ 0.2585 \\ 0.9780 \\ 0.4668 \\ 0.7097 \\ 0.9053 \\ 0.8353 \\ 0.6998 \end{bmatrix}$	$\begin{bmatrix} 0.5754 \\ 0.8850 \\ 0.5136 \\ 0.6945 \\ 0.1099 \\ 0.4552 \\ 0.7095 \\ 0.9782 \\ 0.5892 \\ 0.6749 \\ 0.6672 \\ 0.7165 \\ 0.4854 \end{bmatrix}$

Table 3: **Nonfundamentalness and Invertibility Checks for the SW Model.** Order of shocks: technology, preference, government spending, investment specific, monetary policy, price and wage markup, inflation objective and measurement errors for output growth, consumption growth, investment growth, real wage growth and inflation

As before, the models are solved and simulated through Theorem 1(a) and the conversion procedure set out in Appendix B. We find that the original system with the original sets of measurements and shocks is completely invertible according to Theorem 2, the eigenvalue measures and indeed produces exactly the same simulated moments across the perfect and imperfect information assumptions. As expected, when we add the additional shock in Case 2, compared to non-invertibility of PI the eigenvalues are larger for II ($\mathbb{F}^{II} > \mathbb{F}^{PI}$), introducing non-fundamentalness into the model. The overall fit for fundamentalness under II is much improved from the baseline results (the RBC model), but with a larger-sized model (e.g. Case 2) the difference between PI and II is less marked. Based on Theorem 3 again, this means that the differences between IRFs for PI and II, from the perspective of identifying VARs, are less marked. This result clearly depends on the size of the model and the number of shocks, and via simulation, is consistent with what the literature has seen before, for example, in the empirical exercise of Levine *et al.* (2012b), the estimated NK model with the minimum amount of frictions produces the most notable differences between IRFs when assuming imperfect information for the agents.

In line with the empirical literature again, when we further add measurement errors to the measurement equations for the 4 real variables and the inflation (Case 3), the multivariate fit for fundamentalness or invertibility of SW significantly declines for the both II and PI cases. It is very clear that, even with a medium-sized model like SW, it is the decreasing ratio of observables to shocks that drives a bigger wedge between PI and II, in the sense that the fundamentalness problem worsens for the performance of VARs, and the difference of empirical likelihood between PI and II models increases, with fewer observations by the agents.

7 Conclusions

Our paper has highlighted the choice of information assumptions of agents in a DSGE model as an important source of non-fundamentalness Our general conclusion is that validating a DSGE model by comparing its impulse response functions with those of a data VAR is more problematic when we drop the assumption that agents have perfect information

Our first main result, Theorem 2 is on invertibility, and we have shown that limited information invertibility for the econometrician when agents have perfect information is not necessary and sufficient for invertibility when agents have that same limited information set.

Our second main result, Theorem 3, related to the innovations process with innovations $\hat{\epsilon}_t$. This is relevant when one compares a data VAR with a VAR estimated from *artificial data* generated from our assumed DGP, the DSGE model with fundamental shocks ϵ_t. If the DSGE model is *not* invertible then the latter VAR is the *innovations process* and the reduced form VAR errors are $u_t = Q\hat{\epsilon}_t$, assuming the correct identification matrix Q, and not $u_t = Q\epsilon_t$ where ϵ_t are the fundamental shocks. Our Theorem 3 shows that if the DGP under II is not invertible, then the process for $\hat{\epsilon}_t$ is the same as under PI and of lower order that the process for ϵ_t. This provides an insight into why the innovations process that is the VAR estimated by artificial data cannot generate the impulse response functions of the DSGE model under II.

We develop measures of approximate fundamentalness for both perfect and imperfect information cases. Our results are illustrated using a simple analytical example and we also provide numerical examples for an RBC model and the Smets-Wouters (2007) NK model. We have shown in Theorem 1, we believe for the first time, that a standard rational expectations can always be cast in Blanchard-Kahn form. Finally, we have shown how our results impact on estimation under imperfect information, again when agents and the econometrician have the same information set.

This paper lies with the tradition pioneered by Sims (1980) on the estimation-identification of SVARS. A more recent approach uses "external instruments" which are variables correlated with a particular shock of interest, but not with the other shocks. External instruments can be used to directly estimate causal effects by direct IV regressions using the method of local projections (LP) of Jorda (2005). This method does not require invertibility. Stock and Watson (2018) compares the LP-IV approach with a more efficient SVAR-IV approach proposes a new test for invertibility. It would be of interest to reexamine this method in the light of the information assumptions of agents in the assumed DSGE DGP. Also as mentioned in footnote 4, our analysis can be generalized to allow for agents with different imperfect information observables m_t^A as studied in Lubik *et al.* (2017). These topics will be the subject for future research.[14]

[14]Our imperfect information solution is currently available in Dynare and Bayesian estimation with II and fundamentalness tests are forthcoming.

References

Adam, K. (2007). Optimal Monetary Poicy with Imperfect Common Knowledge. *Journal of Monetary Economics*, **54**(2), 267–361.

Alessi, L., Barigozzi, M., and Capasso, M. (2011). Non-Fundamentalness in Structural Econometric Models: A Review. *International Statistical Review*, **79**(1), 16–47.

Anderson, D. and Moore, J. (1979). *Optimal Filtering*. Dover.

Anderson, G. (2008). Solving linear rational expectations models: A horse race. *Computational Economics*, **31**, 95–113.

Baxter, B., Graham, L., and Wright, S. (2011). Invertible and non-invertible information sets in linear rational expectations models. *Journal of Economic Dynamics and Control*, **35**(3), 295–311.

Beaudry, P., Fève, P., Guay, A., and Portier, F. (2016). When is Nonfundamentalness in SVARs A Real Problem? TSE Working Papers 16-738, Toulouse School of Economics (TSE).

Blanchard, O. and Kahn, C. (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, **48**, 1305–1313.

Brockett, R. W. and Mesarovic, M. D. (1965). The reproducibility of multivariable systems. *Journal of Mathematical Analysis and Applications*, **11**, 548–563.

Canova, F. and Sahneh, M. H. (2017). Are Small-Scale SVARs Useful for Business Cycle Analysis? Revisiting Non-Fundamentalness. *Journal of Economic Dynamics and Control*.

Canova, F., Ferroni, F., and Matthes, C. (2014). Choosing The Variables To Estimate Singular Dsge Models. *Journal of Applied Econometrics*, **29**(7), 1099–1117.

Collard, F. and Dellas, H. (2004). The new keynesian model with imperfect information and learning. *mimeo, CNRS-GREMAQ*.

Collard, F. and Dellas, H. (2006). Misperceived money and inflation dynamics. *mimeo, CNRS-GREMAQ*.

Collard, F., Dellas, H., and Smets, F. (2009). Imperfect information and the business cycle. *Journal of Monetary Economics*, **56**, S38–S56.

Fernandez-Villaverde, J., Rubio-Ramirez, J., Sargent, T., and Watson, M. W. (2007). ABC (and Ds) of Understanding VARs. *American Economic Review*, **97**(3), 1021–1026.

Forni, M., Gambetti, L., and Sala, L. (2016). VAR Information and the Empirical Validation of DSGE Models. Center for Economic Research (RECent), Working Paper 119, University of Modena and Reggio E., Dept. of Economics.

Gali, J. (1999). Technology, Employment and the Business Cycle: Do technology Shocks Explain Aggregate Fluctuations? *American Economic Review*, **89**(1), 249–271.

Giacomini, F. (2013). The relationship between VAR and DSGE models . In T. B. Fomby, L. Kilian, and A. Murphy., editors, *VAR Models in Macroeconomics, Financial Econometrics, and Forecasting - New Developments and Applications: Essays in Honor of Christopher A. Sims*, volume 32 of *Advances in Econometrics*. Emerald Group Publishing Limited.

Graham, L. and Wright, S. (2010). Information, heterogeneity and market incompleteness. *Journal of Monetary Economics*, **57**(2), 164–174.

Ilut, C. and Saijo, H. (2016). Learning, confidence and business cycles. Unpublished Manuscript.

Jorda, O. (2005). Estimation and Inference of Impulse Responses by Local Projections. *American Economic Review*, **95**(1), 161–82.

Levine, P., Pearlman, J., Perendia, G., and Yang, B. (2012a). Endogenous Persistence in an estimated DSGE Model Under Imperfect Information. *Economic Journal*, **122**(565), 1287–1312.

Levine, P., Pearlman, J., Perendia, G., and Yang, B. (2012b). Endogenous Persistence in an Estimated DSGE Model under Imperfect Information. *Economic Journal*, **122**(565), 1287 – 1312.

Lippi, M. and Reichlin, L. (1994). VAR analysis, nonfundamental representations, blaschke matrices. *Journal of Econometrics*, **63**(1), 307–325.

Lubik, T. A., Mertens, E., and Matthes, C. (2017). Indeterminacy and Imperfect Information. 2017 Meeting Papers 337, Society for Economic Dynamics.

Lungu, L., Matthews, K., and Minford, A. (2008). Partial Current Information and Signal Extraction in a Rational Expectations Macroeconomic Model: A Computational Solution. *Economic Modelling*, **25(2)**, 255–273.

Luo, Y. and Young, E. R. (2009). Rational Inattention and Aggregate Fluctuations. *B.E. Journal of Macroeconomic*, **9**.

Mackowiak, B. and Wiederholt, M. (2009). Optimal Sticky Prices under Rational Inattention. *American Economic Review*, **99**(3), 769–803.

Mackowiak, B. and Wiederholt, M. (2011). Business Cycle Dynamics under Rational Inattention. ECB working Paper No 1331.

Minford, A. and Peel, D. (1983). Some Implications of Partial Information Sets in Macroeeconomic Models Embodying Rational Expectations. *Manchester School*, **51**, 225–249.

Neri, S. and Ropele, T. (2012). Imperfect information, real-time data and monetary policy in the euro area. *Economic Journal*, **122**, 651–674.

Nimark, K. (2008). Dynamic Pricing and Imperfect Common Knowledge. *Journal of Monetary Economics*, **55**, 365–382.

Nimark, K. P. (2014). Man-Bites-Dog Business Cycles. *American Economic Review*, **104**(8), 2320–67.

Paciello, L. (2012). Monetary Policy and Price Responsiveness to Aggregate Shocks under Rational Inattention. *Journal of Money, Credit and Banking*, **44**(7), 1375–1399.

Pearlman, J., Currie, D., and Levine, P. (1986). Rational Expectations Models with Private Information. *Economic Modelling*, **3**(2), 90–105.

Sims, C. (2005). Rational Inattention: A Research Agenda. Deutche Bundesbank, W.P. no. 34/2005.

Sims, C. (2011). *Rational Inattention and Monetary Economics*. Elsevier. Chapter in the Handbook of Monetary Economics.

Sims, C. A. (1980). Macroeconomics and Reality. *Econometrica*, **48**(1), 1–48.

Sims, C. A. (2002). Solving Linear Rational Expectations Models. *Computational Economics*, **20**(1–2), 1–20.

Smets, F. and Wouters, R. (2007). Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. *American Economic Review*, **97(3)**, 586–606.

Soccorsi, S. (2016). Measuring nonfundamentalness for structural VARs. *Journal of Economic Dynamics and Control*, **71**(C), 86–101.

Stock, J. and Watson, M. (2018). Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments. *The Economic Journal*, pages 917–948.

Appendix

A Transformation of Model to Blanchard-Kahn Form

A.1 The Problem Stated

The only general results on imperfect information solutions to rational expectations models date back to Pearlman *et al.* (1986), who utilize the Blanchard-Kahn setup, which is given by

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{A.1}$$

with agents' measurements given by

$$m_t = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \tag{A.2}$$

and these can be solved together to yield a reduced-form system. The latter can then be processed via the Kalman filter to obtain the likelihood function for estimation purposes.

Note that measurement errors on observations can be incorporated into ε_t.

Dynare does not accept models in the form of (A.2). In linearized form, the typical Dynare modfile setup will lead to a system of the form

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + \Psi \varepsilon_t \tag{A.3}$$

with measurements

$$m_t = L Y_t \tag{A.4}$$

The next section describes a completely novel algorithm for converting the state space (A.3), (A.4) under partial information to the form (A.2), (A.2). We assume that the system is 'proper', by which we mean the matrix A_1 is invertible; this precludes the possibility of a system that includes equations of the form $h^T Y_{t+1} = 0$, but it is fairly easy to take account of these as well.

A.2 Conversion to Pearlman *et al.* (1986) Setup

Although a little complicated, the basic stages for the conversion are fairly simple:

- Find the singular value decomposition for A_0, so that one can re-define forward-looking variables that are linear combinations of the original Y_t, to yield the smallest number of equations that involve forward-looking expectations.

- Replace any forward-looking expectations that use model-consistent updating equations, which reduces the number of equations with forward-looking expectations.[15] This may need to be repeated a finite number of times. In the case of perfect information this is all that is needed, apart from defining what are the $t+1$ variables.

- For imperfect information, we retain the same backward and forward looking variables as in the perfect information case, but the solution process is a little more intricate.

The procedure for conversion to a form suitable for filtering is as follows:

[15] Suppose $x_t = \rho x_t + \varepsilon_t$, $y_t = E_t x_{t+1}$. Then x_t appears as a forward-looking (FL) variable, and the backward-looking variable (BL) y_t does not appear in a BL equation. However using the first equation we can write $E_t x_{t+1} = \rho E_t x_t$, so that $y_t = \rho E_t x_t$, thereby eliminating the FL equation, and adding a BL equation.

1. (SVD and partitions of A_0) Obtain the singular value decomposition for matrix A_0: $A_0 = U_0 S_0 V_0^T$, where U_0, V_0 are unitary matrices. Assuming that only the first m values of the diagonal matrix S_0 are non-zero ($m = FL\_RANK =$ the rank of S_0), we can rewrite this as $A_0 = U_1 S_1 V_1^T$, where U_1 are the first m columns of U_0, S_1 is the first $m \times m$ block of S_0 and V_1^T are the first m rows of V_0^T. In addition, U_2 are the remaining $n - m$ columns of U_0, and V_2^T are the remaining $n - m$ rows of V_0^T (A_0 is $n \times n$).

2. (Transform (A.3) to FL subsystem using S_1 and U_1) Multiply (A.3) by $S_1^{-1} U_1^T$, which yields:

$$V_1^T Y_{t+1,t} + S_1^{-1} U_1^T A_1 Y_t = S_1^{-1} U_1^T A_2 Y_{t-1} + S_1^{-1} U_1^T \Psi \varepsilon_t \tag{A.5}$$

   Now define forward-looking $x_t = V_1^T Y_t$, backward-looking $s_t = V_2^T Y_t$, and use the fact that $I = VV^T = V_1 V_1^T + V_2 V_2^T$ to rewrite (A.5) as (note that $Y_t = V_1 x_t + V_2 s_t$):

$$x_{t+1,t} + S_1^{-1} U_1^T A_1 (V_1 x_t + V_2 s_t) = S_1^{-1} U_1^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + S_1^{-1} U_1^T \Psi \varepsilon_t \tag{A.6}$$

   or simply:

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.7}$$

   where $F_1 = S_1^{-1} U_1^T A_1 V_1$, $F_2 = S_1^{-1} U_1^T A_1 V_2$, $F_3 = S_1^{-1} U_1^T A_2 V_1$, $F_4 = S_1^{-1} U_1^T A_2 V_2$ and $F_5 = S_1^{-1} U_1^T \Psi$

3. (Transform (A.3) to BL subsystem using U_2) Multiply (A.3) by U_2^T which yields:

$$U_2^T A_1 Y_t = U_2^T A_2 Y_{t-1} + U_2^T \Psi \varepsilon_t \tag{A.8}$$

   which can be rewritten as

$$U_2^T A_1 (V_1 x_t + V_2 s_t) = U_2^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T \Psi \varepsilon_t \tag{A.9}$$

   or more simply:

$$C_1 x_t + C_2 s_t = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.10}$$

   where $C_1 = U_2^T A_1 V_1$, $C_2 = U_2^T A_1 V_2$, $C_3 = U_2^T A_2 V_1$, $C_4 = U_2^T A_2 V_2$ and $C_5 = U_2^T \Psi$.

If C_2 is invertible then multiply (A.10) by C_2^{-1}, and go straight to Stage 6. If C_2 is not invertible, then write (A.7) and (A.10) in the more general form:

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 \varepsilon_t \tag{A.11}$$

$$C_1 x_t + C_2 s_t + G_1 x_{t,t} + G_2 s_{t,t} = C_3 x_{t-1} + C_4 s_{t-1} + C_5 \varepsilon_t \tag{A.12}$$

We do this because the next stage may have to be iterated a finite number of times.

4. (<u>$C_2 + G_2$ singular</u>) Find, a matrix J_2 such that $J_2^T(C_2 + G_2)$=0 (by using the SVD of $C_2 + G_2$). Then take forward expectations of (A.12) and pre-multiply by J_2^T to yield:

$$J_2^T(C_1 + G_1)x_{t+1,t} = J_2^T C_3 x_{t,t} + J_2^T C_4 s_{t,t} \tag{A.13}$$

Then reduce the number of forward-looking variables by substituting for $x_{t+1,t}$ from (A.11). In addition find a matrix Q that has the same number of columns as $J_2^T(C_1 + G_1)$ and is made up of rows that are orthogonal to it. Then we define

$$\begin{bmatrix} \bar{x}_t \\ \hat{x}_t \end{bmatrix} = \begin{bmatrix} Q \\ J_2^T(C_1 + G_1) \end{bmatrix} x_t \qquad x_t = M_1 \bar{x}_t + Q_2 \hat{x}_t \tag{A.14}$$

where $[Q_1 \;\; Q_2] = \begin{bmatrix} Q \\ J_2^T(C_1 + G_1) \end{bmatrix}^{-1}$ From the substitution of $x_{t+1,t}$ into (A.13), we can rewrite the system in terms of forward-looking variables $\bar{x}_t$ and backward-looking variables s_t, $\hat{x}_t$:

$$\bar{x}_{t+1,t} + QF_1Q_1\bar{x}_t + [QF_2 \;\; QF_1Q_2] \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{A.15}$$
$$= QF_3Q_1\bar{x}_{t-1} + [QF_4 \;\; QF_3Q_2] \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + QF_5\varepsilon_t$$

$$\begin{bmatrix} C_1Q_1 \\ J_2^T(C_1 + G_1)F_1Q_1 \end{bmatrix} \bar{x}_t + \begin{bmatrix} C_2 & C_1Q_2 \\ J_2^T(C_1 + G_1)F_2 & J_2^T(C_1 + G_1)F_1Q_2 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{A.16}$$

$$
+ \begin{bmatrix} G_1Q_1 \\ J_2^T C_3 Q_1 \end{bmatrix} \bar{x}_{t,t} + \begin{bmatrix} G_2 & G_1Q_2 \\ J_2^T C_4 & J_2^T C_3 Q_2 \end{bmatrix} \begin{bmatrix} s_{t,t} \\ \hat{x}_{t,t} \end{bmatrix}
$$

$$
= \begin{bmatrix} C_3Q_1 \\ J_2^T (C_1+G_1)F_3Q_1 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} C_4 & C_3Q_2 \\ J_2^T (C_1+G_1)F_4 & J_2^T(C_1+G_1)F_3Q_2 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix}
$$

$$
+ \begin{bmatrix} C_5 \\ J_2^T (C_1+G_1)F_5 \end{bmatrix} \varepsilon_t
$$

The number of forward-looking states has decreased because $\bar{x}_t = Q_1 x_t$, and the number of backward-looking states $\bar{s}_t = \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix}$ has increased by the same amount. In addition the relationship $Y_t = V_1 x_t + V_2 s_t$ has changed to

$$
Y_t = V_1 Q_1 \bar{x}_t + \begin{bmatrix} V_2 & V_1 Q_2 \end{bmatrix} \bar{s}_t \tag{A.17}
$$

The system is now again the form of (A.11) and (A.12). Repeat this stage until $C_2 + G_2$ is of full rank.

Proof of Theorem 1(a). In the perfect information case, the form (A.11), (A.12) with $s_t = s_{t,t}, x_t = x_{t,t}$ is generated after a finite number of iterations of Stage 3 - the number of iterations cannot exceed the number of variables. The forward looking variables are now x_t and the backward looking variables are s_t and x_{t-1}, and the system can be set up in Blanchard-Kahn form by defining $z_{t+1} = \begin{bmatrix} s_t \\ x_t \end{bmatrix}$. The only additional calculation is to invert $C_2 + G_2$ to obtain the equation for s_t, and to substitute into (A.11). ■

From this point, we eschew the details of matrix manipulations, as these are much more straightforward to understand conceptually compared with those above.

5. (<u>C_2 non-singular</u>) Firstly form expectations of (A.12), and invert $C_2 + G_2$ to obtain $s_{t,t}$ in terms of $x_{t,t}, x_{t-1,t}, s_{t-1,t}, \epsilon_{t,t}$. Then substitute this back into (A.12), and invert C_2 to yield an expression for s_t in terms of the above expected values and also $x_t, x_{t-1}, s_{t-1}, \epsilon_t$. This can be further substituted into (A.11) to yield an expression for $x_{t+1,t}$ in terms of these variables and their expectations. Similarly the measurement

equations $m_t = LY_t$ can now be expressed in terms of all these variables. It follows that if we define $z_{t+1} = \begin{bmatrix} \epsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, then the system can now be described by (A.2).

6. (C_2 singular) We again start from (A.11) and (A.12), and regard x_t as the forward looking variable and s_t, x_{t-1} as the backward looking variables. Now advance these equations by changing t to $t+k: \ k = 1, 2, 3, ...$ and take expectations using information at time t, implying that $E_t s_{t+k} = E_t s_{t+k,t+k}$. Because $C_2 + G_2$ is invertible, we can rewrite these equations with just $x_{t+k+1,t}$ and $s_{t+k,t}$ on the LHS. Then the usual Blanchard-Kahn conditions for stable and unstable roots imply a saddlepath relationship of the form

$$x_{t+k+1,t} + N_1 s_{t+k,t} + N_2 x_{t+k,t} = 0 \tag{A.18}$$

where $[I \ \ N_1 \ \ N_2]$ represents the eigenvectors of the unstable eigenvalues. In particular, this holds for $k = 0$, so if we substitute for $x_{t+1,t} = -N_1 s_{t,t} - N_2 x_{t,t}$ into (A.11), then together with (A.12) we obtain solutions for x_t, s_t in terms of $x_{t,t}, s_{t,t}, x_{t-1}, s_{t-1}, \epsilon_t$. This is possible, because we have assumed the system is proper i.e. A_1 is invertible[16], and any manipulations of A_1 in the previous stages have been simple linear transformations of it to yield the matrices F_1, F_2, C_1, C_2. In addition, when we take expectations of (A.12) at time t, given that $C_2 + G_2$ is invertible, we obtain an equation for $s_{t,t}$ in terms of $x_{t,t}, s_{t-1,t}, x_{t-1,t}, \epsilon_{t,t}$. It therefore follows that we can write s_t is terms of these latter variables as well as the variables above (excluding $s_{t,t}$). The same will be true of the the measurements $m_t = LY_t$.

At this point we have expressions for x_t and s_t, without any effect from $x_{t+1,t}$, so in principle we could solve the signal processing problem from this point onwards. However for consistency with the case of C_2 nonsingular, we can retrieve the representation of $x_{t+1,t}$ by substituting for s_t back into (A.11), and then the system has the same structure as that for the case C_2 nonsingular.

[16]The algorithm can be reworked without too much much difficulty if for example some of the forward looking equations in (A.3) are of the form $S_0 E_t Y_{t+1} = 0$.

Finally, to summarise the required Blanchard-Kahn setup described by (A.2)

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1}$$

and we define $z_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \end{bmatrix}$, the converted form (A.2) becomes (when invertibility of A_0 holds)

$$\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ FF_4 & FF_3 & FF_2 & FF_1 \\ 0 & 0 & 0 & 0 \\ FF_8 & FF_7 & FF_6 & FF_5 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} \quad \text{(A.19)}$$

where $G_{13} = -C_2^{-1}C_1$, $G_{12} = C_2^{-1}C_3$, $G_{11} = C_2^{-1}C_4$, $P_1 = C_2^{-1}C_5$, $G_{33} = -F_2G_{13} - F_1$, $G_{32} = -F_2G_{12}+F_3$, $G_{31} = -F_2G_{11}+F_4$, $P_3 = -F_2P_1+F_5$, $FF_1 = -C_2^{-1}G_1+C_2^{-1}G_2(C_2+G_2)^{-1}(C_1+G_1)$, $FF_2 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_3$, $FF_3 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_4$, $FF_4 = -C_2^{-1}G_2(C_2+G_2)^{-1}C_5$, $FF_5 = -F_2FF_1$, $FF_6 = -F_2FF_2$, $FF_7 = -F_2FF_3$ and $FF_8 = -F_2FF_4$. The C and F matrices are the reduction system matrices in (A.15) and (A.16) in the form of (A.11) and (A.12) (i.e. the iterative procedure that ensures invertibility to be achieved).

The measurements $m_t = LY_t$ can be written in terms of the states as $m_t = L(V_1x_t + V_2s_t)$, where V_1, V_2 have been updated by (A.17) through the same reduction procedure as above. Using (A.19), we show that m_t can be rewritten as

$$\begin{aligned} m_t &= \begin{bmatrix} LV_2P_1 & LV_2G_{11} & LV_2G_{12} & LV_1 + LV_2G_{13} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} \\ &+ \begin{bmatrix} LV_2FF_4 & LV_2FF_3 & LV_2FF_2 & LV_2FF_1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t,t} \\ s_{t-1,t} \\ x_{t-1,t} \\ x_{t,t} \end{bmatrix} \end{aligned} \tag{A.20}$$

So the observations (A.20) can now be cast into the form in (A.2)

$$m_t = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix}$$

where $M_1 = [LV_2P_1 \ LV_2G_{11} \ LV_2G_{12}]$ and $M_2 = LV_1 + LV_2G_{13}$. Similarly, $M_3 = [LV_2FF_4 \ LV_2FF_3 \ LV_2FF_2]$ and $M_4 = LV_2FF_1$. Thus the setup is as required, with the vector of predetermined variables given by $[\varepsilon_t' \ s_{t-1}' \ x_{t-1}']'$, and the vector of jump variables given by x_t.

B Extending the Sims Solution to the Imperfect Information Case

Sims (2002) sets up the model in the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi\eta_t \tag{B.1}$$

where y_t includes and forward-looking expectations, and η_t satisfy $E_t\eta_{t+1} = 0$; Γ_0 is in general singular. He then computes a QZ decomposition for Γ_0, Γ_1 such that the unstable part of the system is given by Z_2y_t, which satisfies

$$\Lambda_{22}Z_2y_t = \Omega_{22}Z_2y_{t-1} + Q_2(\Psi\varepsilon_t + \Pi\eta_t) \tag{B.2}$$

where Λ_{22} is in general singular. $Z_2 y_{t-1}$ is solved forwards in time; perfect information then implies

$$E_{t-1} Z_2 y_{t-1} = E_t Z_2 y_{t-1} (= Z_2 y_{t-1}) \tag{B.3}$$

which in turn implies that

$$Q_2(\Psi \varepsilon_t + \Pi \eta_t) = 0 \text{ and } Z_2 y_{t-1} = 0 \tag{B.4}$$

Thus $Z_2 y_t = 0$ represents the saddlepath relationship. Furthermore, assuming that the terms in η_t in the rest of the system defined by the QZ decomposition are linearly dependent on $Q_2 \Pi$, it is then easy to solve for the remaining transformed states of the system as a vector autoregression in ε_t.

For the *imperfect information case*, (B.3) no longer holds. If we assume that η_t is known at time t, it follows that $E_{t-1} Z_2 y_{t-1} = 0$, but

$$E_t Z_2 y_{t-1} = -\Omega_{22}^{-1} Q_2(\Psi E_t \varepsilon_t + \Pi \eta_t) \tag{B.5}$$

It therefore follows that the remaining states $Z_1 y_t$ will be dependent on $Z_1 y_{t-1}$, ε_t and in addition $E_t Z_2 y_{t-1}$ and $Q_2 \Psi E_t \varepsilon_t$, so that the overall solution will be as complicated as that derived by Pearlman *et al.* (1986). In particular, one has to define the updating equation for $E_t Z_2 y_{t-1}$ in terms of $E_{t-1} Z_2 y_{t-1}$ and the observations at time t, and solve for this in dynamic equilibrium.