Asset Price Learning And Optimal Monetary Policy

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Abstract

We characterize optimal monetary policy when agents are learning about endogenous asset prices, but are close to rational otherwise. Boundedly rational expectations induce inefficient equilibrium asset price fluctuations which translate into inefficient aggregate demand fluctuations. We find that the optimal policy raises interest rates when expected capital gains, and the level of current asset prices, is high. The optimal policy does not eliminate deviations of asset prices from their fundamental value. When monetary policymakers are information-constrained, optimal policy can be reasonably approximated by simple interest rate rules that incorporate capital gains. Our results are robust to alternative belief specifications.

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1 Introduction

Should monetary policy react to asset prices? This question has been the subject of a longstanding debate in monetary economics. Asset price misalignments, including

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bubbles, might pose risks to macroeconomic and financial stability. However, standard macroeconomic models rule out asset price misalignments by design, and are therefore of limited use in addressing these concerns. when misalignments are caused by rational bubbles, Gali (2014, 2017) shows that if anything, nominal interest rates should be lowered in the presence of a bubble. In this paper, we show that the opposite is true when misalignments are instead caused by boundedly rational asset price expectations.

We argue that non-rational expectations in financial markets create a need for monetary policy to react to asset prices. We consider an environment in which agents hold subjective beliefs about future asset prices and update these beliefs through learning. We show that, while flexible inflation targeting remains optimal under learning, implementing optimal policy requires the policy rate to react to asset prices in addition to inflation and the output gap. The key to the mechanism is that under subjective beliefs asset prices influence the natural rate of interest, independent of their correlation with technology and real allocation. This creates a need for the monetary authority to "lean against the wind". Simple interest rate rules which include a weighted average of past asset prices do a good job of replicating optimal policy when the central bank cannot observe private subjective beliefs.

Learning has recently emerged as a plausible explanation for many well-known asset price characteristics (Fuster *et al.* 2012; Collin-Dufresne *et al.* 2013; Adam *et al.* 2015; Barberis *et al.* 2015 for stock prices; Adam *et al.* (2012); Caines (2016); Glaeser and Nathanson (2017) for house prices). Subjective beliefs about future prices/returns are recursively updated to account for new information, and most of the observed variation in asset prices is explained not by changes in discount rates but instead by changes in subjective expectations about future returns. Survey measures of return expectations are typically found at odds with rational expectations, but are consistent with learning models (Greenwood and Shleifer, 2014). While the effects of learning on asset price behavior have been discussed previously, we view these effects of also being of straightforward concern for policymakers.

The model we use to analyze policy is stylized, but familiar. It is the standard New-Keynesian model, to which we add a durable asset and learning about the equilibrium asset price. In our baseline version of the model, agents' subjective belief about the future growth rate of prices is updated in an extrapolative fashion using past observation, as in Adam *et al.* (2017). The more agents have seen asset prices appreciate in the recent past, the more they expect positive price growth in the future as well. While this is our preferred specification of beliefs, we also show that our results carry through to more general settings of behavioral or near-rational beliefs such as the natural expectations of Fuster *et al.* (2012). We also extend our results to the case of asset production, which adds an investment margin to the model.

In the presence of nominal rigidities, non-rational asset price expectations affect perceived household wealth, causing fluctuations in aggregate demand. This is the only channel through which asset price misalignments affect allocations in our model. We are aware that there are other, potentially more important channels through which asset prices can affect the real economy, most importantly credit frictions (Bean, 2004). We see the wealth effect channel in this paper as a stand-in for these more complex transmission channels of asset prices. The advantage of this simplification is that we are able to obtain closed form solutions for optimal monetary policy in the presence of learning.

In the absence of cost push shocks, the flexible price allocation is first-best efficient. It is well known that under rational expectations, strict inflation targeting is then optimal. We find that it continues to be optimal under learning, but that the implementation requires a very different interest rate policy: The optimal interest rate leans against the wind as it is increasing in the level of asset prices, as well as in agents' beliefs about future house price growth.

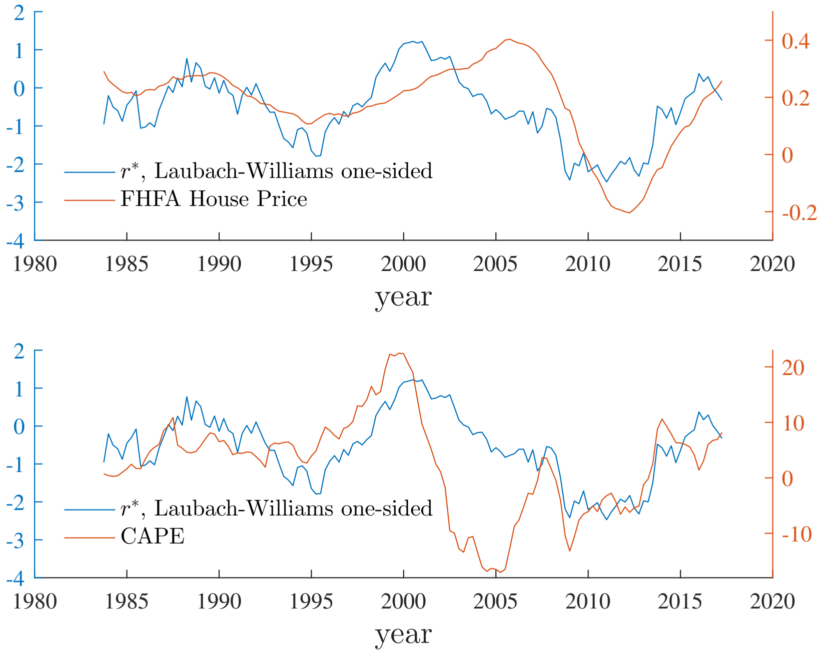
The key to understand this result is to understand that the natural real rate—the real interest rate that would prevail in the absence of nominal rigidites—depends on expectations. In standard models, rational expectations are determined by equilibrium allocations, so that the natural real rate is determined by technology and preferences alone. Under learning, however, agents' subjective expectations about asset prices affect the natural real rate: If agents expect larger capital gains on the durable asset, then the expected real returns on debt must also rise for the bond market to clear, whether the capital gains expectations are warranted or not. When prices are rigid, setting the interest rate below this "perceived natural rate" raises aggregate demand and inflation as agents try to frontload the future windfalls from subjectively expected capital gains.

Because subjective asset price growth is extrapolated from observed prices in our model, the dependence of the natural real rate asset price expectations implies comovement between the natural rate and realized asset prices. This is indeed a future of US data, and can be seen in Figure 1, which plots the 20-quarter change in the well-known Laubach and Williams (2016) natural rate estimate against the 20-quarter change in both the FHFA house price index for the United States as well as the Shiller CAPE ratio.[1]

When cost push shocks are added to the model, it is possible to replicate the outcome

[1] We thank Kevin Lansing for pointing this relationship out to us.

Figure 1: 20-Quarter Change in the Natural Rate of Interest & Asset Prices.



of optimal monetary policy under rational expectations under learning. Again, the implementation of this outcome is different, by letting the interest rate fluctuate around the perceived natural rate, leading to a nominal interest rate that leans against the wind. In a numerical exercise, we find that simple interest rate rules that incorporate a reaction to a moving average of past asset price changes perform well in the learning model.

Formulating a non-rational expectations equilibrium requires one to explicitly spell out the entire belief formation process. That is, one must specify precisely how agents form expectations about future income, inflation, interest rates and so on. Here, we isolate the effects of asset price learning from other channels through which learning influences the economy. We do so by restricting expectations to be model-consistent conditional on the subjective asset price belief, as in Winkler (2016).[2] This restriction implies a minimal departure from rational expectations as agents' expectations, conditioned on future asset prices, remain consistent with model outcomes. Even with this restriction, asset price learning has real effects in the model, allowing for feedback effects between the financial and the real side of the economy.

Our paper relates to a long-standing literature that analyses leaning against the wind. Standard DSGE models with rational expectations usually do not support such policies or find very small benefits at best (Bernanke and Gertler (2001); Faia and Mona-

[2]The concept has also been applied in Caines (2016) and Gandré (2017).

celli (2007); Curdia and Woodford (2010), to name just a few). In these models there is typically little reason for asset price stabilization, beyond that achieved in the course of stabilizing inflation and the output gap, or asset price volatility is caused by rational bubbles that start and end independently of policy actions.[3]

As mentioned in the beginning, our paper is closely related to Gali (2017), who studies a New-Keynesian model with overlapping generations and allows for stationary, rational asset price bubbles. Some of his findings are mirrored in this paper, such as the desirability of strict inflation targeting when the only source of fluctuations are aggregate demand shocks. However, asset prices in our ,pde; deviate from fundamentals in our framework due to boundedly rational expectations, which is quite different from a rational bubbles model. Raising interest rates slows down asset price growth in our model, while it speeds up price growth in the rational bubble of Gali (2014). Policies that lean against the wind are therefore more effective when asset price misalignments arise from learning as opposed to rational bubbles.

Our analysis also shares some features with Christiano *et al.* (2010), who study the optimal policy reaction to news shocks about future productivity. In their model, news shocks cause asset prices to rise, and also increase the natural real rate of interest, so that monetary policy should optimally respond by raising interest rates. In this paper, the natural real rate fluctuates in response to endogenous changes in subjective expectations that can be entirely independent of productivity.

A number of studies, including Dupor (2005), Gilchrist and Saito (2009), and Mertens (2011), have argued that monetary policy should react to asset prices in environments that depart from rational expectations. In these papers, distortions to beliefs about asset prices have real effects through investment channels. Monetary policy that reacts to asset prices can then counteract inefficeint fluctuations in real allocations. In this paper we present an argument for leaning against the wind, even in the absence of allocative inefficiencies of this kind. Our results continue to hold if we include asset production in the model.

Our paper adds to a small number of studies deriving optimal policy in macroeconomic models with learning. Fully optimal policy has recently been studied in a two-equation model with learning by Molnar and Santoro (2014) and Eusepi and Preston (2016). While these studies are concerned with learning about inflation and marginal

[3] An exception is Airaudo *et al.* (2013) and Dong *et al.* (2017). In Airaudo *et al.* (2013) asset prices can be subject to subspots, and a reaction of interest rates to asset prices eliminates the subspot equilibria. Dong *et al.* (2017) study liquidity bubbles in the presence of collateral constraints and numerically optimize over optimal interest rate rule coefficients, finding that under certain conditions a reaction to asset prices increases welfare. The optimal reaction is to cut the interest rate when the bubble is expanding, as if to “lean into the wind”, consistent with the intuition of Gali (2017).

costs, here we focus on learning about asset prices. This serves to isolate the implications for expectations-drive asset price fluctuations on monetary policy, as opposed to other channels through which learning can affect the economy. Airaudo (2016) aguments the standard New Keynesian model with a stock market and infinite-horizon learning (as in Preston (2006)) to study conditions under which the rational expectations equilibrium is learnable, but stops short of characterizing optimal policy.[4]

The remainder of this paper is structured as follows. We begin by describing the model in Section 2, and our notion of a learning equilibrium in Section 3. We characterize the linearized equilibrium under rational expectations and learning in Section 4. Optimal policy is analyzed in Sections 5 and 7, while Section 6 discusses how well certain simple interest rate rules approximate the optimal policy. Section 9 concludes.

2 Model description

Our model is a standard New-Keynesian model in which the representative household also holds a stock of an asset that yields utility. The supply of the asset in the economy is fixed. One can think of the asset as a stock of housing, but we will refer to it as a generic durable asset. Because the asset is in fixed supply, the dynamics of inflation and output are unaffected by its presence under rational expectations. Under learning, however, we will get non-trivial effects of asset prices on allocations.

We first describe the model for a general description of expectations. A representative household provides labor and owns firms. It can also hold nominal bonds promising a nominal return i_t. In addition, the household owns the durable asset. The household's problem is

$$\mathbb{E}^{\mathscr{P}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} + \chi \frac{H_t^{1-\theta}}{1-\theta} \right)$$

$$\text{s.t. } C_t \quad = \quad W_t N_t + \Pi_t + T_t - Q_t (H_t - H_{t-1}) + B_t - \frac{1+i_{t-1}}{1+\pi_t} B_{t-1}.$$

Here, C_t is the household's utility from consuming final consumption goods, N_t is the household's labor supply, and T_t are lump-sum taxes. Π_t are the profits received from

[4]Outside of the learning literature, Gabaix (2016) develops a particular form of myopia and studies optimal policy under commitment. In his model, agents have attenuation bias and underestimate the persistence of economic fluctuations. In this paper, agents instead overestimate their persistence, at least along the dimension of asset prices. As a result, the optimal policy needs to react aggressively to the extrapolative bias, while Gabaix finds that policy can afford to be less aggressive in the presence of attenuation bias.

firms. The quantity of the asset owned by the household is denoted H_t and trades at the price Q_t. B_t are government bonds which are in zero net supply. The price level is P_t and $\pi_t = P_t/P_{t-1} - 1$ is the inflation rate. The expectational operator $\mathbb{E}^{\mathscr{P}}$ has a superscript indicating that agents' expectations are evaluated under a subjective probability measure $\mathscr{P}$ that need not coincide with rational expectations.

The first order conditions are:

$$
\begin{aligned}
W_t &= C_t^{\gamma} N_t^{\phi} \\
1 &= \beta \mathbb{E}_t^{\mathscr{P}} \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} \frac{1+i_t}{1+\pi_{t+1}} \\
Q_t &= \chi \frac{C_t^{\gamma}}{H_t^{\theta}} + \beta \mathbb{E}_t^{\mathscr{P}} \left(\frac{C_t}{C_{t+1}} \right)^{\gamma} Q_{t+1}.
\end{aligned}
$$

On the production side, a representative intermediate goods producer transforms household labor into intermediate goods using the decreasing returns to scale technology

$$\tilde{Y}_t = A_t N_t^{\alpha}.$$

It has to hire labor at the real wage rate w_t and sells its goods at the real price M_t. Its first-order condition is

$$W_t = \alpha M_t A_t N_t^{\alpha - 1}.$$

There is a contiuum of wholesalers indexed by $i \in [0,1]$ who transform the undifferentiated intermediate good into differentiated goods using a one-for-one technology. They face a standard Dixit-Stiglitz demand function and a Calvo price setting friction. When producer i is able to set a price P_{it} for its output Y_{it}, it solves:

$$\max_{P_{it}} \mathbb{E}_t^{\mathscr{P}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^{s} \xi \Lambda_{t,t+\tau} \right) \left((1+\tau_t) P_{it} - M_{t+s} P_{t+s} \right) Y_{it+s}$$

$$\text{s.t. } Y_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\sigma} Y_t,$$

where σ is the demand elasticity of substitution between varieties, $\Lambda_{t,t+\tau} = \beta^{\tau} C_t^{\gamma} C_{+\tau}^{-\gamma}$ is the household discount factor between times t and $t+\tau$, ξ is the probability of not being able to adjust the price in the future, τ_t is a time-varying government subsidy to revenue. The subsidy is set such that it eliminates the mark-up distortions in the steady state, but shocks to the subsidy will act as cost-push shocks. Any profits are distributed to households. The first-order conditions are standard and give rise to the New-Keynesian Phillips curve.

A representative retailer buys differentiated goods from wholesalers at the price P_{it} and transforms them back into a homogenous final consumption good. The final good sells at price P_t and is produced according to the technology

$$Y_t = \left(\int_0^1 (Y_{it})^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

The first order condition gives rise to a constant elasticity of substitution σ between varieties. The price level can be expressed as $P_t = \int_0^1 P_{it} Y_{it} / Y_t$.

The government transfers a lump sum real amount to households

$$T_t = \tau_t \int_0^1 P_{it} Y_{it} di$$

to finance the subsidies to final good producers and offset the tax on stock holdings. Profits and government transfers sum up to $\Pi_t + T_t = Y_t - W_t N_t$. Finally, the central bank sets the nominal interest rate.

We allow for productivity and cost-push shocks which follows first-order autoregressive processes:

$$\begin{aligned} \log A_t &= (1-\rho_a) \log \bar{A} + \rho_a \log A_{t-1} + \varepsilon_{at} \\ \tau_t &= (1-\rho_\tau) \bar{\tau}_t + \rho_\tau \tau_{t-1} + \varepsilon_{\tau t} \end{aligned}$$

The innovations are independent white noise with variances σ_A^2 and σ_τ^2.

Market clearing in the final goods market requires $Y_t = C_t$. Bonds are in zero net supply and the market clearing condition is therefore $B_t = 0$. Finally, the supply of the durable is fixed at unity, so that asset market clearing requires $H_t = 1$.

3 Equilibrium

The equilibrium under rational expectations is standard—it is the New Keynesian model. The asset price Q_t is redundant because the durable asset is in fixed supply. Agents with rational expectations never expect to buy or sell their holdings of the asset, not because they expect to be unable to to do so—they are price takers in a competitive market in which they are able to buy and sell as they please—but because they expect the price in every state of the world to be such that they will not ever want to buy or sell.

Let's recall the formal definition of a rational expectations equilibrium. Let $y_t \in \mathbb{R}^N$ denote the collection of all endogenous model variables—including prices, allocations, and strategies—and by $u_t \in \mathbb{R}^M$ the collection of all exogenous model variables at time t, which I will call "fundamentals". Stochastic processes for y_t and u_t are defined on the spaces $\Omega_y = \Pi_{t=0}^{\infty}\mathbb{R}^N$ and $\Omega_u = \Pi_{t=0}^{\infty}\mathbb{R}^M$, respectively. Further, denote by $\Omega_u^{(t)}$ the set of all possible histories of exogenous variables up to period t, and its elements by $u^{(t)} \in \Omega_u^{(t)}$. Finally, let $\mathbb{P}_u$ denote the true probability measure for the exogenous variables defined on $(\Omega_u, \mathscr{S}(\Omega_u))$, where $\mathscr{S}(\cdot)$ is the Borel sigma-algebra on a metric space. The topological support of $\mathbb{P}_u$ is denoted by $\text{supp}(\mathbb{P}_u)$.

Definition 1. A *rational expectations equilibrium* is a sequence of mappings $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \ldots$ such that, for all t and $u^{(t)} \in \text{supp}(\mathbb{P}_u)$:

1. the choices contained in y_t solve the time-t decision problem of each agent in the economy, conditional on decision-relevant[5] past and current outcomes contained in $u^{(t)}$ and $y^{(t)} = \left(g_0\left(u^{(0)}\right), \ldots, g_t\left(u^{(t)}\right)\right)$, and evaluating the probability of future external decision-relevant outcomes under the probability measure $\mathbb{P}$ implied by $\mathbb{P}_u$ and the mappings $(g_t)_{t=0}^{\infty}$;

2. the allocations contained in $y_t = g_t\left(u^{(t)}\right)$ clear all markets.

Under learning, we assume that agents are not endowed with the knowledge of the equilibrium asset price process. Intuitively, they do not know whether there is a representative household investing in the asset, or whether other investors are trading in unknown ways that cause seemingly random price fluctuations. Faced with this lack of knowledge, agents forecast prices using a subjective belief system. As we show in Section 7, this subjective belief system can be made quite general, but here we confine ourselves to our preferred specification which follows Adam *et al.* (2017). Agents think that the asset price is a simple random walk model with a time-varying drift:

$$\Delta \log Q_t = \mu_{t-1} + \varepsilon_t \tag{3.1}$$

$$\mu_t = \rho_\mu \mu_{t-1} + \nu_t. \tag{3.2}$$

where the shocks $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ and $\nu_t \sim \mathcal{N}\left(0, \sigma_\nu^2\right)$ are iid white noise, independent of the rest of the economy. Since these two shocks are not observable, agents have to

[5] A variable is *decision-relevant* if it enters the agents' decision problem, and a decision-relevant variable is *external* if its value is taken as given by the agent, while it is *internal* if the variable is part of the solution of the agents' decision problem. For example, investors need to get information on current and future dividends set by firms (decision-relevant and external to investors) to decide themselves on optimal stock holdings (decision-relevant and internal), while they do not need to forecast wages since they do not receive labor income (not decision-relevant).

use the Kalman filter to form a belief about the hidden state μ_t. The asset price can equivalently be written just in terms of observables and the filtered state:

$$\Delta \log Q_t = \hat{\mu}_{t-1} + z_t \tag{3.3}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t \tag{3.4}$$

Here, $\hat{\mu}_t$ is the belief about μ_t ; g is the weight agents place on new data when updating their beliefs, which is a function of the perceived variances of ε_t and ν_t; and $z_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$ is the forecasting error in the filtering problem. This forecasting error is exogenous normally distributed white noise under agents' subjective expectation, the variance of which is decreasing in the signal-to-noise ratio $\sigma_\nu^2/\sigma_\varepsilon^2$. In order to avoid complications arising from simultaneity in the determination of outcomes and beliefs, we follow Adam *et al.* (2012) and Caines (2016) and assume that in period t agents make choices conditional on $\hat{\mu}_{t-1}$, and update their beliefs according to (3.3) at the end period.

In order to determine expectations about the remaining variables of the model, including inflation, we follow Winkler (2016) in assuming that agents have so-called "conditionally model-consistent expectations". This is a restriction on expectations that effectively allows us to isolate the effects of asset price learning from other potential sources of learning in the economy. Conditionally model-consistent expectations are consistent with all equilibrium conditions of the model, except those that would convey knowledge of the price that clears the asset market.

Formally, let $(\Omega_z, \mathcal{S}(\Omega_z), \mathcal{P}_z)$ be the probability space that defines the subjective beliefs for z_t (i.e., the z_t are iid normally distributed with mean zero and variance σ_z^2). Agents' subjective beliefs depend on this perceived stochastic forecast error even though in equilibrium, model outcomes are a function only of fundamentals u_t. The subjective probability measure $\mathcal{P}$ is defined by a mapping from fundamentals u_t and the subjective forecast error z_t to model outcomes y_t.

Definition 2. *Conditionally model-consistent expectations* (CMCE) are a sequence of mappings $h_t : \Omega_u^{(t)} \times \Omega_z^{(t)} \ni \left(u^{(t)}, z^{(t)}\right) \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \ldots$ such that, for all t and $\left(u^{(t)}, z^{(t)}\right) \in \operatorname{supp}\left(\mathcal{P}_{u,z}\right)$:

1. the choices contained in y_t solve the time-t decision problem of each agent in the economy, conditional on decision-relevant past and current outcomes contained in $u^{(t)}$ and $y^{(t)} = \left(h_0\left(u^{(0)}, z^{(0)}\right), \ldots, h_t\left(u^{(t)}, z^{(t)}\right)\right)$, and evaluating the probability of future decision-relevant outcomes under the probability measure $\mathcal{P}$ implied by $\mathbb{P}_u \otimes \mathcal{P}_z$ and the mappings $(h_t)_{t=0}^{\infty}$;

2. the allocations contained in $y_t = h_t\left(u^{(t)}, z^{(t)}\right)$ clear all markets except the markets for assets and final consumption goods;

3. asset prices under $\mathscr{P}$ follow the law of motion given by (3.3)–(3.4).

The definition of the mappings h_t defining expectations is almost identical to the definition of a rational expectations equilibrium, except that asset market equilibrium is not part of the conditions, and instead the price Q_t evolves according to subjective beliefs. Conditional model consistency restricts the subjective belief $\mathscr{P}$ to have the maximum degree of consistency with the model given agents' misspecified belief about asset prices. In analogy to the adaptive learning literature, we call the mappings h_t defining beliefs the *perceived law of motion (PLM)*.

Computing the learning equilibrium is an easy two-step procedure: First, compute the PLM $h_t\left(u^{(t)}, z^{(t)}\right)$; second, compute the ALM $g_t\left(u^{(t)}\right)$. Both steps are no more complicated than solving the rational expectations equilibrium.

While under $\mathscr{P}$, demand for the durable asset does not have to be equal to supply, in equilibrium the market still has to clear:

Definition 3. An *equilibrium* with conditionally model-consistent expectations is a sequence of mappings $r_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto z_t \in \mathbb{R}$ and $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \dots$ such that, for all t and $u^{(t)} \in \operatorname{supp}(\mathbb{P}_u)$:

1. $g\left(u^{(t)}\right) = h\left(u^{(t)}, \left(r_0\left(u^{(0)}\right), \dots, r_t\left(u^{(t)}\right)\right)\right)$;

2. the allocations contained in $y_t = g_t\left(u^{(t)}\right)$ clear the asset market.

Market clearing is brought about by finding the right value of the price Q_t that clears the housing market. We call the resulting equilibrium mappings g_t the *actual law of motion (ALM)*. This mapping implies a particular path for z_t, the subjective house price forecast error. In equilibrium, z_t will be a function of the states and the shocks of the model, while under $\mathscr{P}$ it is perceived as an unforecastable exogenous disturbance. It is precisely in this way that $\mathscr{P}$ violates the rational expectations hypothesis.

Agents endowed with conditionally model-consistent expectations may not know the equilibrium pricing function, but they make the smallest possible expectational errors consistent with their subjective view about the evolution of stock prices. This way of setting up expectations is very tractable and can be readily applied in a variety of models, as we have shown in previous papers (Caines, 2016; Winkler, 2016). It also allows us to transparently solve the linearized version of the model.

4 Linearized equilibrium

The analysis in this paper will focus entirely on the linearized version of the model. The conditionally model-consistent expectations under learning imply that the learning equilibrium can be linearized in much the same way as its rational expectations counterpart.

4.1 Rational expectations equilibrium

Under rational expectations, we can linearize around the non-stochastic steady state to obtain:

$$y_t = a_t + \alpha n_t \tag{4.1}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi} \left(w_t - a_t + (1-\alpha) n_t\right) + \eta_t \tag{4.2}$$

$$w_t = \gamma y_t + \phi n_t \tag{4.3}$$

$$i_t = \gamma \left(\mathbb{E}_t y_{t+1} - y_t\right) + \mathbb{E}_t \pi_{t+1} \tag{4.4}$$

$$q_t = \gamma y_t - \beta\gamma \mathbb{E}_t y_{t+1} + \beta \mathbb{E}_t q_{t+1}. \tag{4.5}$$

Here, lower-case variables denote log-linearizations around the zero-inflation steady state, except for i_t which is the difference of the nominal interest rate from its steady-state level, and $\eta_t = \frac{(1-\xi)(1-\beta\xi)}{\xi}\left(\bar{\tau} - \tau_t\right)$ is the cost-push shock process. The model is simply the textbook New-Keynesian model with an extra equation for the price of housing. Note that it still has to be closed with an equation describing the conduct of monetary policy, such as an interest rate rule. Note that asset holdings h_t are known to be constant in equilibrium. As a result, h_t does not even enter the rational expectations equilibrium conditions.

An important special case of the model obtains when prices are fully flexible and there are no cost-push shocks ($\xi = 0$ and $\eta_t = 0$). In this case, the allocation in the rational expectations equilibrium is first-best efficient everywhere. Output and the real interest rate are independent of the conduct of monetary policy and are given by:

$$y_t^{n,RE} = \frac{\phi + 1}{\phi + 1 - \alpha\left(1-\gamma\right)} a_t \tag{4.6}$$

$$r_t^{n,RE} = -\frac{\gamma\left(\phi+1\right)}{\phi + 1 - \alpha\left(1-\gamma\right)} \left(1 - \rho_a\right) a_t. \tag{4.7}$$

These quantities are also called the natural level of output and the natural real rate, respectively.

As is well known, the equilibrium with sticky prices can be expressed in terms of the deviation from this efficient equilibrium. To this end, denote the output gap by $\hat{y}_t = y_t - y_t^{n,RE}$. The sticky price equilibrium can be summarized with the standard two equations:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + \eta_t \tag{4.8}$$

$$\mathbb{E}_t \hat{y}_{t+1} - \hat{y}_t = \frac{1}{\gamma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^{n,RE}\right). \tag{4.9}$$

where $\kappa = (1-\xi)\left(1-\beta\xi\right)\left(1+\phi-\alpha+\alpha\gamma\right)/\xi\alpha$. These equations are the standard New-Keynesian Phillips curve and the IS equation.

4.2 Learning equilibrium

In order to solve the learning equilibrium, we have to proceed in two steps. The first is to solve for the agents' policy functions given their beliefs $\mathscr{P}$; the second is to impose market clearing in the asset market to back out the equilibrium asset price.

It is easy to verify that the equilibrium under learning has the same non-stochastic steady state as the rational expectations equilibrium, and we take this as our linearization point. The expectations of agents as well as their optimal choices under the subjective measure $\mathscr{P}$ are expressed as the solution to a "perceived law of motion" (PLM) that, in its linearized form, consists of the following equations:

$$y_t = a_t + \alpha n_t \tag{4.10}$$

$$\pi_t = \beta \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi}\left(w_t - a_t + (1-\alpha)\, n_t\right) + \eta_t \tag{4.11}$$

$$w_t = \gamma c_t + \phi n_t \tag{4.12}$$

$$i_t = \gamma\left(\mathbb{E}_t^{\mathscr{P}} c_{t+1} - c_t\right) + \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} \tag{4.13}$$

$$q_t = \gamma c_t - \left(1-\beta\right)\theta h_t - \beta\gamma \mathbb{E}_t^{\mathscr{P}} c_{t+1} + \beta \mathbb{E}_t^{\mathscr{P}} q_{t+1} \tag{4.14}$$

$$c_t = y_t - \frac{\bar{Q}\bar{H}}{\bar{Y}}\left(h_t - h_{t-1}\right) \tag{4.15}$$

$$q_t = q_{t-1} + \hat{\mu}_{t-1} + z_t \tag{4.16}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t \tag{4.17}$$

This system can be solved as if it were a rational expectations model. However, under $\mathscr{P}$, there is an additional shock, the asset price forecast error z_t, that is absent under rational expectations. This shock will be predictable in equilibrium, but under under $\mathscr{P}$ agents believe it to be unforecastable. The first stage of the solution has to take this into account. Just as under rational expectations, one still needs to add an equation describing the conduct of monetary policy, such as an interest rate rule, for the above system to be fully determined.

Having solved for expectations and optimal choices given $\mathscr{P}$, the equilibrium under learning (also called "actual law of motion" or ALM) is then found by imposing market clearing in the market for housing. This condition simply reads

$$h_t = 0.$$

This equation implicitly defines the equilibrium realizations of z_t. Contrary to agents' beliefs, this forecast error is not an exogenous shock in equilibrium, but an endogenous variable. It is precisely in this sense that expectations are not rational in this model.

4.2.1 Flexible price PLM

As before, we first describe the flexible price equilibrium ($\xi = 0$ and $\eta_t = 0$). We find the flex-price PLM by solving (4.10)–(4.17) under subjective beliefs $\mathscr{P}$. The learning model has two additional state variables compared to its rational expectations counterpart, q_t and $\hat{\mu}_{t-1}$. We guess and verify that the asset demand function has the following form:

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} - k_q q_t + k_\mu \hat{\mu}_{t-1} \tag{4.18}$$

where the coefficients satisfy $k_h \in (0,1)$, $k_a, k_q, k_\mu > 0$. Exact expressions can be found in the appendix. Asset demand under learning is increasing in productivity, decreasing in the asset price, and increasing in expectations of future asset price growth.

We can also solve for the values of output and the real interest rate under flexible prices. Under subjective expectations, these are functions of the fundamental, last period's asset holdings, the asset price which is perceived as exogenous, and price growth expectations. Below, we write output and the real rate in deviation from their rational expectations counterpart:[6]

$$y_t^{n,PLM} = y_t^{n,RE} + \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha}\left(k_a a_t - (1-k_h)\, h_{t-1}^{n,PLM} - k_q q_t + k_\mu \hat{\mu}_{t-1}\right) \tag{4.19}$$

$$\begin{aligned} r_t^{n,PLM} &= r_t^{n,RE} + \gamma\kappa_1 \left(k_a \left(2-\rho_a - k_h\right) a_t - (1-\kappa_h)^2\, h_{t-1}^{n,PLM} - k_q\,(1-k_h)\, q_t\right) \\ &\quad + \gamma\kappa_1 \left(\left(2-\rho_\mu - k_h\right) k_\mu + k_q\right) \hat{\mu}_{t-1}. \end{aligned} \tag{4.20}$$

where the constant κ_1 is

$$\kappa_1 = \frac{1+\phi-\alpha}{1+\phi-\alpha\left(1-\gamma\right)} \frac{\bar{Q}\bar{H}}{\bar{Y}} > 0.$$

This natural rate is increasing in the price growth belief $\hat{\mu}_{t-1}$. Agents' subjective expectations about output under flexible prices are affected by the choice of asset holdings (which are not constant in agents' minds). An increase in expected asset price growth will increase asset demand, and households will increase their labor supply in order to finance their purchase of the asset, thereby increasing the level of output.

The natural real rate under subjective expectations can be understood by the arbitrage relationship between the return on the durable asset and the return on bonds. Combining the two asset pricing equations (4.13) and (4.14), we obtain:

$$r_t^{n,PLM} = \frac{1-\beta}{\beta}\left(\gamma c_t - \theta h_t - q_t\right) + \mathbb{E}_t^{\mathcal{P}} q_{t+1} - q_t$$

The expected return on the two assets has to be equal up to first order. An increase in expected durable asset price growth $\hat{\mu}_{t-1} = \mathbb{E}_t^{\mathcal{P}} q_{t+1} - q_t$ increases the expected return to the durable asset, and the real interest rate on bonds therefore has to rise as well.

4.2.2 Flexible price ALM

To find the flex-price equilibrium under learning, i.e. the actual law of motion, one has to impose $h_t = 0$. From the housing demand function (4.18), one can then immediately solve for the equilibrium asset price and the realization of the subjective forecast error:

$$\begin{aligned} 0 &= k_a a_t - k_q q_t + k_\mu \hat{\mu}_{t-1} \\ \Leftrightarrow q_t &= \frac{k_a a_t + k_\mu \hat{\mu}_{t-1}}{k_q}. \end{aligned} \tag{4.21}$$

[6] Since the PLM includes asset holdings h_t as an endogenous state variable, there are two possible definitions of a natural real rate and natural level of output (Neiss and Nelson, 2003; Woodford, 2003). One can either define them as conditional on the actual level of asset holdings h_{t-1} under sticky prices, or as conditional on the level of asset prices $h_{t-1}^{n,PLM}$ that would obtain had prices been flexible in the past as well, given the history of exogenous shocks. Here, we opt for the latter definition.

That is, the equilibrium asset price is increasing in both productivity and house price growth expectations. This is intuitive. The demand function (4.18) is downward-sloping, and so an increase in demand due to either higher productivity (i.e. higher income) or higher expected capital gains has to be met with an increase in the price to bring about equilibrium in the asset market.

It becomes clear that the forecast error z_t is everything but unforecastable:

$$z_t = \frac{1}{k_q} k_a a_t - q_{t-1} + \left(\frac{k_\mu}{k_q} - 1 \right) \hat{\mu}_{t-1}.$$

This is precisely the way in which rational expectations break in this model. If agents had the correct belief about z_t, then owing to their conditionally-model consistent expectations, their beliefs would be correct and their expectations would be rational.

Substituting the equilibrium price (4.21) into Equations (4.19) and (4.20), we obtain the realized level of output and the real rate under flexible prices:

$$y_t^{n,ALM} = y_t^{n,RE} \tag{4.22}$$
$$r_t^{n,ALM} = r_t^{n,RE} + \gamma \kappa_1 \left(k_a \left(1 - \rho_a \right) a_t + \left(\left(1 - \rho_\mu \right) k_\mu + k_q \right) \hat{\mu}_t \right). \tag{4.23}$$

Under learning and flexible prices, the equilibrium level of output is the same as under rational expectations. This coincidence arises because, under flexible prices, output is determined entirely by intratemporal conditions that are independent of expectations. Nonetheless, the real interest rate does depend on expectations, and its natural level under learning is therefore different from rational expectations. In particular, it is increasing in subjectively expected house price growth.

4.2.3 Sticky prices

Just as under rational expectations, the sticky price equilibrium under learning can be expressed in deviation from the flexible price allocation, which greatly helps our analysis. Denote by $\hat{h}_t = h_t - h_t^{n,PLM}$ the difference of asset holdings from their flex-price level under the PLM, which we call the "asset gap". Next, denote by $\hat{c}_t = h_t - h_t^{n,PLM}$ the difference of consumption from its flex-price level, which we call $\hat{c}_t$ the "consumption gap". Then the sticky price equilibrium in the PLM can be summarized with three equations:

$$\pi_t = \beta \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} + \kappa \left(\hat{c}_t + \kappa_1 \Delta \hat{h}_t \right) + \eta_t \tag{4.24}$$

$$\hat{c}_t = \mathbb{E}_t^{\mathscr{P}} \hat{c}_{t+1} - \frac{1}{\gamma} \left(i_t - \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} - r_t^{n,PLM} \right) \tag{4.25}$$

$$\hat{h}_t = \frac{\gamma}{\theta \left(1 - \beta\right)} \left(\hat{c}_t - \beta \mathbb{E}_t^{\mathscr{P}} \hat{c}_{t+1} \right) \tag{4.26}$$

The first equation is the familiar Phillips curve, but where the output gap is replaced by the consumption gap, augmented by asset purchases $\Delta \hat{h}_t$.[7] The intuition is that asset purchases are financed by an increase in labor supply, which drives down firms' marginal cost of production and therefore reduces inflation. The second equation is the familar IS equation, where again the output gap is replaced by the consumption gap since the two are not equal under agents' subjective expectations. The third equation is the Euler equation for housing demand, rewritten in gap form.

Notice that the asset price q_t itself does not appear in the Euler equation for asset holdings in gap form. Agents perceive it to be an exogenous process, and therefore to be independent of the degree of price stickiness. It therefore drops out of the gap between flexible and sticky prices, and what is left are those variations in asset demand that are due to variations in the household's discount factor. The asset price still implicitly enters equation (4.26) through the natural rate $r_t^{n,PLM}$.

To find the actual law of motion under sticky prices, one imposes $h_t = h_t^{n,PLM} + \hat{h}_t = 0$ and solves for q_t. The equilibrium depends crucially on the behavior of the nominal interest rate i_t, which we have not specified yet.

4.3 Numerical illustration

We illustrate the properties of the learning model using a simple and standard calibration. We set the capital share in output, α, equal to 0.7 and the discount factor β equal to 0.995. The coefficient of relative risk aversion, γ, and the elasticity of labor supply, $1/\phi$, are set to 1.1 and 0.33, respectively. The parameter governing the price elasticity of demand for consumption varieties is set to $\sigma = 4$, while the Calvo pricing frictions are set to $\kappa = 0.75$. We set the learning gain to $g = 0.03$. This parameter governs the degree of expected predictability in asset price changes, and a higher gain is associated with stronger equilibrium house price volatility. The chosen value is within the range

[7] Even though in equilibrium (in the ALM) there can be no housing purchases as the supply of housing is fixed, one cannot simply set $\Delta h_t = 0$ to compute the equilibrium, since agents are not aware of this restriction.

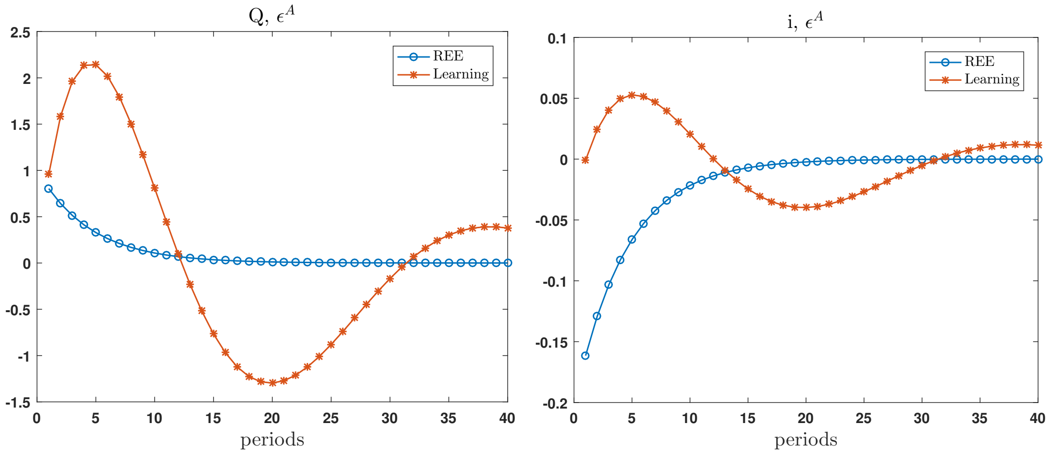
considered in the learning literature (e.g. Adam *et al.*, 2017). Finally, we follow Billi (2017) and set the autocorrelation of both the technology and cost-push shocks to 0.8 and their standard deviations to 0.008.

We first document the effect of learning under flexible prices. Here, learning has no effect on equilibrium allocations relative to rational expectations, but manifests itself only in the realized asset price and interest rate process. Figure 2 plots the response of asset prices Q_t and the real interest rate i_t to a technology shock ϵ_A under rational expectations and learning. The effect of learning on Q_t is typical for self-referential asset price learning models. Initially, the asset price Q_t rises on impact because higher wage wage income raises asset demand, as under rational expectations. But the initial increase now causes a subsequent revision in beliefs $\hat{\mu}_t$ through the learning mechanism. The household believes that the shock has some long-run impact on house price growth and responds by increasing its demand for housing above the rational expectations demand. This response drives a further increase in Q_t in the next period and the shock continues to propagate through belief updating thereafter. At some point, expected price growth has risen so much that it outstrips realized price growth. At this point, beliefs $\hat{\mu}_t$ decrease, bringing about a reduction in housing demand and therefore in equilibrium house prices, so that the process eventually reverts back to steady state.

The differing response of the interest rate between the learning and rational expectations environments directly shows the effect of expected asset price growth on the natural rate of interest. Even though the impulse response of *realized* consumption is exactly identical under learning and rational expectations, what matters for the interest rate is *expected* consumption growth. Under learning, increases in expected asset price growth in the periods following the shock also increase expected consumption growth, as agents anticipate to sell some of their asset holdings in the future to profit from the capital gains. Higher expected consumption growth implies a higher real rate of interest under the PLM than under rational expectations.

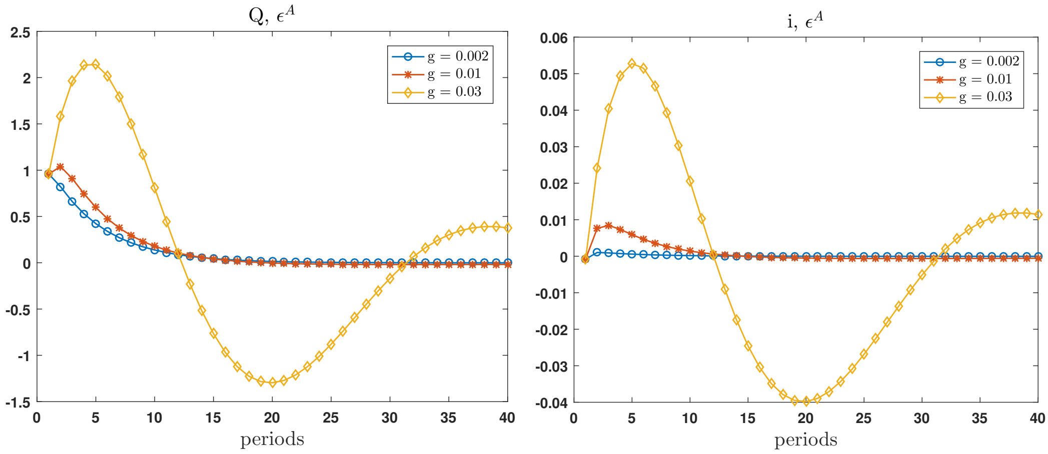
Figure 3 plots the response of Q_t and i_t to a technology shock ϵ_A under learning and flexible prices for different values of the gain parameter, g. A larger learning gain implies that expectations of future asset price growth respond more strongly to shocks, causing larger equilibrium price fluctuations in turn. For a gain close to zero (the blue circled line in the figure), equilibrium house prices are very close to their rational expectations counterpart. The response of the interest rate, however, remains very different from rational expectations. As the learning gain increases, the asset price as well as the interest rate become more volatile.

Figure 2: Effect of learning under flexible prices.



Note: Response to a one standard deviation positive technology shock ε_{At}. Log percentage points. Flexible prices and zero inflation.

Figure 3: Role of the learning gain g, flexible prices.



Note: Response to a one standard deviation positive technology shock ε_{At}. Log percentage points. Flexible prices and zero inflation.

5 Optimal Policy

5.1 Welfare functions

We provide second-order approximations to the expected discounted sum of utility in our model. Under learning, an important distinction has to be made whether welfare is evaluated under the subjective law of motion (in which asset supply is variable and prices are a random walk) or under the actual law of motion (in which asset supply is fixed and prices are functions of the model fundamentals).

If we evaluate welfare under the actual law of motion, then welfare takes the form

$-\sum_{t=0}^{\infty}\mathbb{E}_0\mathscr{L}_t$, where the period loss function is given by

$$\mathscr{L}_t = 2\sigma\frac{\xi}{1-\xi}\frac{1}{1-\beta\xi}\pi_t^2 + \left(\gamma + \frac{1-\alpha+\phi}{\alpha}\right)\left(y_t - y_t^{n,RE}\right)^2. \tag{5.1}$$

This loss function is identical to that of the standard rational expectations New Keynesian model. It penalizes deviations of inflation from zero as well as deviations of output from the natural rate of output under rational expectations (4.6). This natural rate of output is first-best efficient under the ALM.

By contrast, the welfare function under the PLM takes a quite different form. Welfare is approximated by $-\sum_{t=0}^{\infty}\mathbb{E}_0^{\mathscr{P}}\mathscr{L}_t^{PLM}$, where the period loss function is given by

$$\begin{aligned}\mathscr{L}_t^{PLM} &= 2\sigma\frac{\xi}{1-\xi}\frac{1}{1-\beta\xi}\pi_t^2 + \left(\gamma + \frac{1-\alpha+\phi}{\alpha}\right)\hat{c}_t^2 \\ &+ \frac{1-\alpha+\phi}{\alpha}\left(\frac{\bar{Q}\bar{H}}{\bar{Y}}\Delta\hat{h}_t\right)^2 + 2\left(\gamma+\frac{1-\alpha+\phi}{\alpha}\right)\hat{c}_t\left(\frac{\bar{Q}\bar{H}}{\bar{Y}}\Delta\hat{h}_t\right) \\ &- (1-\beta)\,\theta\frac{\bar{Q}\bar{H}}{\bar{Y}}\hat{h}_t^2.\end{aligned} \tag{5.2}$$

where $\hat{c}_t$ and $\hat{h}_t$ are the deviations of consumption and asset holdings from their PLM-flexible price levels. Here, the period loss takes the form of deviations from the flexible price allocations under the PLM, and also includes terms for asset holdings. Those terms do not appear in the ALM loss function because asset holdings are constant in equilibrium.

5.2 Optimal policy without cost-push shocks

We now solve for the optimal monetary policy, first under the assumption that there are no cost-push shocks. For exposition, we start by reviewing the optimal policy under rational expectations. As the flexible price equilibrium under rational expectations is first-best efficient, monetary policy is optimal if it manages to replicate the flexible price allocation in the presence of nominal rigidities. This amounts to closing the output gap and completely stabilizing the price level at the same time, as can be seen from the loss function (5.1). Without cost-push shocks, the "divine coincidence" holds and complete stabilization is achievable. The optimal policy implements

$$\pi_t = 0.$$

From the Phillips curve (4.8), it immediately follows that $\hat{y}_t = 0$. The optimal policy can be implemented with the following rule:

$$i_t = r_t^{n,RE} + \phi_\pi \pi_t$$

where ϕ_π can be any number satisfying the Taylor principle $\phi_\pi > 1$. The interest rate has to track the natural real rate and react more than one-for-one to inflation, i.e. satisfy the Taylor principle.

Under learning, the question is which welfare criterion to use. Should the central bank aim to maximize agents' subjectively *expected* discounted utility and minimize the loss function 5.2 under the PLM? Or should it aim to maximize average *realized* utility and minimize the loss function 5.1 under the ALM? Fortunately, both welfare criteria prescribe the same optimal outcome here.

Proposition 1. *The optimal monetary policy under learning implements $\pi_t = 0$ and $y_t = y_t^{n,RE}$, regardless of whether welfare is evaluated under the ALM or the PLM. The optimal policy can be implemented with the rule $i_t = r_t^{n,PLM} + \phi_\pi \pi_t$, where $\phi_\pi > 1$.*

Proof. Suppose that the central bank implemented $\pi_t = 0$. The PLM Phillips curve (4.24) then reduces to the relationship

$$\hat{y}_t = \frac{\gamma\alpha}{1+\phi-\alpha\left(1-\gamma\right)} \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta \hat{h}_t.$$

Substituting into the housing demand equation (4.26), we obtain a second-order difference equation of the form

$$\left(1-\beta\right)\frac{\theta}{\gamma}\hat{h}_t = -\kappa_1 \left(\Delta \hat{h}_t - \beta \mathbb{E}_t^{\mathcal{P}} \Delta \hat{h}_{t+1}\right).$$

It is easily verified that the only solution to this equation is $\hat{h}_t = 0$. But this implies that we implement the flexible price allocation. From the subjective perspective of agents, the flexible price allocation is first-best efficient. Therefore, strict inflation targeting is optimal from the subjective perspective of agents. Moreover, the actual equilibrium in this economy has $\pi_t = 0$ and $y_t = y_t^{n,RE}$, as was shown in the last section. This allocation is also first-best efficient under model-consistent expectations, and therefore optimal under model-consistent expectations as well. □

It might seem at first that the presence of learning does not alter the prescriptions of optimal policy because the target criterion strict inflation targeting is unchanged. But the implementation of this target requires a different reaction function under learning.

The nominal interest rate has to track the natural real interest rate $r_t^{n,PLM}$ as agents perceive it under subjective expectations. This natural rate is very different from the one under rational expectations. Whereas $r_t^{n,RE}$ is a function of productivity a_t only, $r_t^{n,PLM}$ depends additionally on beliefs $\hat{\mu}_t$, prices q_t and the asset holdings h_{t-1}. In particular, the real rate rises when expected asset price growth $\hat{\mu}_t$ increases. In equilibrium, the asset price q_t depend positively on expected price growth, and it is in this sense that the optimal monetary policy leans against the wind: In times of high prices, the interest rate has to be high to track the perceived natural real rate.

The equilibrium realization of the nominal rate under the optimal policy is the expression $r_t^{n,ALM}$ derived in (4.23). However, an instrument rule that prescribes $i_t = r_t^{n,ALM} + \phi_\pi \pi_t$ would fail to implement the optimal policy. The equilibrium natural rate $r_t^{n,ALM}$ only coincides with the perceived natural rate when $h_t = 0$. While this must be the case in equilibrium, agents under $\mathscr{P}$ contemplate other possible realizations of the house price for which they plan on choosing $h_t \neq 0$. These off-equilibrium states of the world enter into agents' expectations of future marginal costs. Therefore, the central bank must promise to stabilize inflation even in these off-equilibrium states. Tracking only the equilibrium natural rate is insufficient: It must track the perceived natural rate.

As an illustration, figure 4 shows impulse responses for the learning model with three interest rate equations:

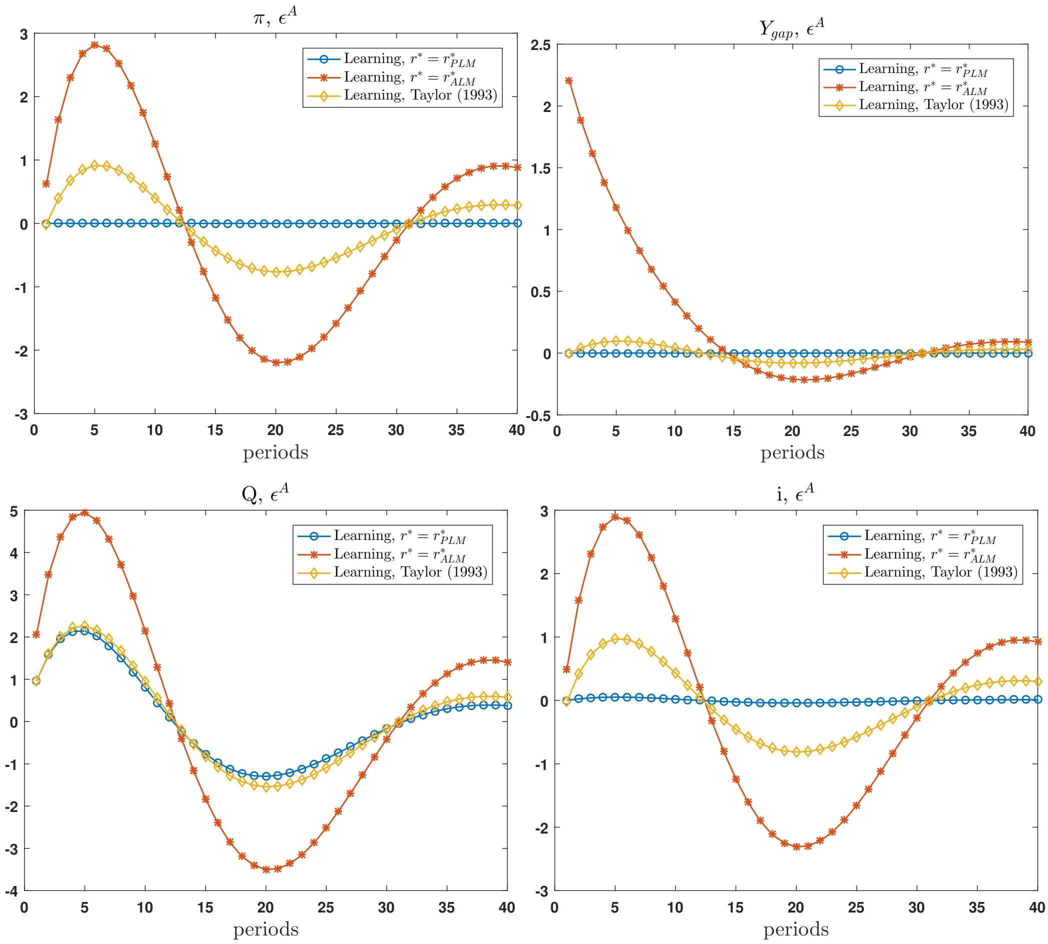
$$i_t = r_t^{n,PLM} + 1.05\pi_t \tag{5.3}$$

$$i_t = r_t^{n,ALM} + 1.05\pi_t \tag{5.4}$$

$$i_t = r_{ss} + 1.5\pi_t + 0.125\hat{y}_t. \tag{5.5}$$

The first equation (5.3) implements strict inflation targeting as per Proposition 1. The only difference of the second equation (5.4) is that the monetary authority reacts to the equilibrium process of the natural rate insteadof the perceived process. Figure 4 shows how using the ALM natural rate of interest in the the policy rule does not yield a zero inflation outcome. Why not? As discussed in the last section, the central bank must promise to stabilize inflation even in those states that are never reached in equilibrium—that is, when the housing market doesn't clear—but contemplated by agents under their subjective expectations. The second of the above policy rules fails to do so. Due do their beliefs about the process governing Q_t, agents under the PLM do not account for the effect of the technology shock on future asset price growth. Consequently, the initial response of consumption is smaller than under rational expectations. From the standpoint of an agent under the flex price ALM on the other hand, the technology shock has an anticipated positive impact on the path of Q_t due to expected asset demand. As a result, the initial consumption response and subsequent consumption decline will be greater. The ALM natural rate of interest declines more upon the impact of the shock than does the PLM natural rate of interest. When a monetary authority uses the ALM natural rate in its policy rule as in (5.4), then, the nominal interest does not increase sufficiently to prevent an inflationary response.

Figure 4: Optimal policy and alternatives.



Note: Response to a unit standard deviation positive technology shock ε_{At} under sticky prices. Log percentage points. The interest rate rules used are given in Equations (5.3)–(5.5).

Finally, the third equation is a standard Taylor rule. Figure 4 shows that this rule performs reasonably well, but does not implement strict inflation targeting. Interestingly, the nominal interest rate is more volatile under the Taylor rule than under the optimal rule (5.3) which reacts to asset prices. The reason is of course that the stabilization benefits of reacting to asset prices make equilibrium nominal rates more stable as well.

5.3 Optimal policy with cost-push shocks

In the presence of cost-push shocks, the so-called divine coincidence does not hold and the first-best allocation is not feasible. Under rational expectations, it is well known that the optimal discretionary policy seeking to minimize the loss function (5.1) satisfies (e.g. Woodford, 2003):

$$\pi_t = \zeta\eta_t \tag{5.6}$$

$$\left(y_t - y_t^{n,RE}\right) = -\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa\zeta}\pi_t \tag{5.7}$$

$$i_t = r_t^{n,RE} + \left(\rho + \gamma\left(1-\rho\right)\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\right)\frac{1}{\zeta}\pi_t \tag{5.8}$$

where the sensitivity ζ of inflation to the cost-push shock is given by $\zeta_{disc} = \left(1-\beta\rho+\frac{\kappa^2}{\lambda}\right)^{-1}$. It is also possible to solve for the weight ζ^* that minimizes the loss function (5.1) within the class of policies implementing $\pi_t = \zeta\eta_t$. This optimal weight is given by $\zeta^* = \left(1-\beta\rho+\frac{\kappa^2}{\lambda}/\left(1-\beta\rho\right)\right)^{-1}$.

We do not solve for optimal discretionary or commitment policy under learning. Our more modest result is that the outcomes of the optimal discretionary policy under RE can be replicated under learning.

Proposition 2. *The allocation in (5.6)–(5.7) is attainable under learning for any value of ζ in the ALM. The nominal interest rate that implements the allocation follows*

$$\begin{aligned} i_t = {} & r_t^{n,PLM} + a\frac{1+\phi-\alpha}{\kappa_1\alpha}\hat{y}_t \\ & + \left(\rho+\gamma\left(1-\rho\right)\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\left(\left(1+\frac{\kappa_1\gamma\left(1-\beta\rho\right)}{\theta\left(1-\beta\right)+\gamma\beta a}\right)^{-1} + a\frac{1+\phi-\alpha}{\kappa_1\alpha\kappa\gamma\left(1-\rho\right)}\right)\right)\frac{1}{\zeta}\pi_t. \end{aligned} \tag{5.9}$$

where the coefficient $a \in (0,1)$ is defined in the appendix.

In particular, the optimal discretionary policy outcome under RE is attainable under learning with $\zeta = \zeta_{disc}$. Moreover, within the class of policies implementing $\pi_t = \zeta\eta_t$, the weight $\zeta = \zeta^*$ also maximizes ALM welfare under learning.

The expression for the nominal interest rate shows that the equilibrium interest rate path depends on inflation and the perceived output gap as well as the natural rate. The natural rate is increasing in the level of asset prices as well as the subjective expectation of future asset price growth. Therefore, the nominal interest rate in (A.5) is effectively leaning against the wind.

6 Simple rules and leaning against the wind

Implementing optimal policy in the learning environment requires knowledge of the natural rate of interest under the PLM. In particular, it implies that the monetary authority knows the agents' beliefs about μ_t, which in part determine $r^{n,PLM}$ in equilibrium. An obvious concern is that policy makers might not observe those beliefs that are subjective and privately held. In this section, we consider what remedies exist for a monetary policymaker in such a situation. More precisely, we show that incorporating asset prices into the monetary policy rule can allow a monetary authority who does not observe beliefs to approximate optimal policy.

We compute the model under the assumption that the monetary authority is following a Taylor-type rule of the form:

$$i_t = r_{ss} + \phi_\pi \cdot \pi_t + \phi_y \cdot \hat{y}_t + \phi_q (1-\tilde{\omega}) \sum_{s=0}^{\infty} \tilde{\omega}^s \Delta \log Q_{t-s}. \tag{6.1}$$

The rule depends on inflation and the output gap, and has an additional term for asset prices: a moving average of past price changes, with a weight on past observations that decays at the rate $\tilde{\omega} \in (0,1)$. In what follows, we keep the coefficient on inflation at $\phi_\pi = 1.5$ and find coefficients on the output gap and asset prices $\left(\phi_y, \phi_q, \omega\right)$ that minimize either (5.1) under the equilibrium probability measure, or (5.2) under the subjective probability measure.[8] We constrain all coefficients to be positive.

Table 1 shows the optimized rule coefficients.

Column (1) shows the optimal coefficient on the output gap under rational expectations when $\phi_q = 0$ while Column (2) shows values optimized jointly over $\left(\phi_y, \phi_q, \tilde{\omega}\right)$. Raising rates when asset prices are rising does not increase welfare, but lowers it (in fact, the optimization would like to set a negative value for ϕ_q).

Under learning, the picture is markedly different. Column (3) shows the optimal coefficient on the output gap when $\phi_q = 0$, while Column (2) shows values optimized jointly over $\left(\phi_y, \phi_q, \omega\right)$. The optimal rule in Column (4) includes a positive reaction to asset price growth, and achieves a lower loss function value than in Column (3). This lower loss results primarliy from a reduction in inflation volatility. Interestingly, the optimal rule does reduce asset price volatility to the level under rational expectations: Asset

[8] If one also optimizes over the coefficient and inflation, then the optimal policy under rational expectations is given by $\phi_\pi \to \infty$ and $\phi_y/\phi_\pi \to \zeta > 0$ (Boehm and House, 2014). The outcomes of this limit policy are also attainable under learning with a similar policy that also responds infinitely strongly to inflation and the ouput gap. In this section, we rule out infinite rule coefficients by keeping the inflation coefficient fixed, and focus only on the tradeoff of reacting to the output gap and asset prices.

Table 1: Optimized simple rule coefficients.

	RE, $\mathscr{L}$		Learning, $\mathscr{L}$		Learning, $\mathscr{L}^{PLM}$	
	(1)	(2)	(3)	(4)	(5)	(6)
ϕ_y	0.209	0.209	0.085	0.032	0.065	0.065
ϕ_q		0		0.948		0
$\tilde{\omega}$		n/a		0.975		n/a
$\sigma(\pi_t)$	0.325	0.325	0.262	0.098	0.259	0.138
$\sigma\left(y_t - y_t^{n,RE}\right)$	0.460	0.460	0.495	0.583	0.524	0.518
$\sigma(q_t)$	0.923	0.923	0.992	1.169	1.051	1.038
Loss fn value	2.923	2.923	2.117	0.909	2.132	0.987

Note: Loss function values and unconditional standard deviation of inflation π_t, house prices q_t and output gap $y_t - y_t^{n,RE}$ in percentage points. Columns (1) and (2) are evaluated under rational expectations, Columns (3) and (4) under learning.

price volatility does not enter the loss function directly, and so the central bank cares only about the effects of asset price movements on inflation and the output gap.

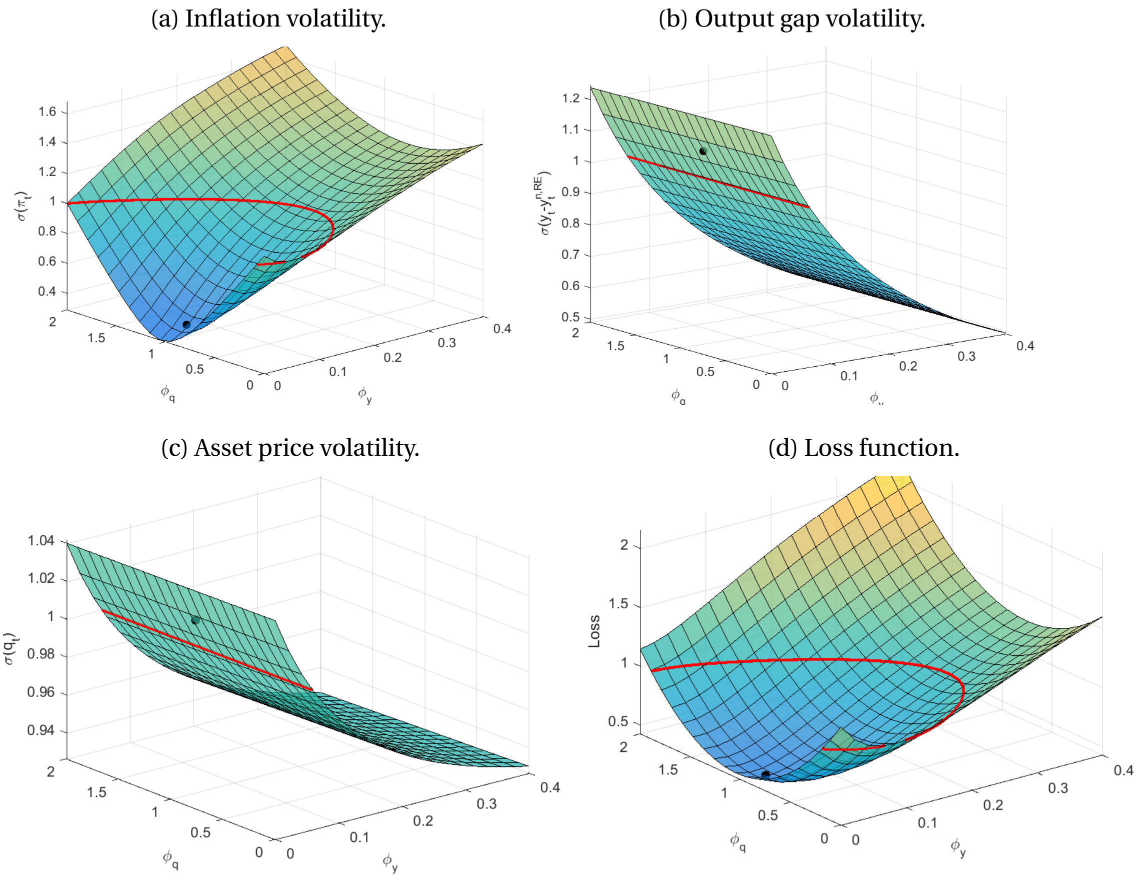
Finally, we also compute in Columns (5) and (6) the rule coeffiients that agents in the learning model would find optimal themselves under the subjective probability measure $\mathscr{P}$. These coefficients are different from what the central bank chooses in order to maximize agents' objective expected utility. In particular, agents do not perceive any gains from a positive interest rate reaction to asset price growth. This illustrates the tension of optimal policy exercises without rational expectations: Prescriptions taken to maximize expected utility depend on which expectations are used.

To get a better idea of the effects of monetary policy reactions to asset prices and the output gap under learning, we compute loss function values as well as the volatilities of inflation, the output gap and asset prices over a range of parameters. We fix the moving average weight to $\tilde{\omega} = 0.975$ as in Column (4) of Table 1 and vary the magnitude of the response coefficients ϕ_y and ϕ_q on the output gap and inflation.[9] Figure 5 contains the results as surface plots.

The effect of changes in the coefficient on the output gap are as expected: They lower the volatility of the output gap itself, but increase the volatility of inflation. This trade-off arises because the model has cost-push shocks in it. A reaction to the output gap also lowers asset price volatility in this model. But importantly, reacting to asset prices also carries stabilization benefits. The volatility of asset prices is slightly decreasing in the asset price response ϕ_q. The volatility of the output gap is also barely affected by the asset price response. But the volatility of inflation can be reduced significantly with $\phi_q > 0$. In sum, the loss function is minimized at a strictly interior point at which the central bank reacts to both the output gap and asset price growth.

[9] Our results are qualitatively robust to changes in the moving average weight ω. In particular, a positive reaction to asset prices $\phi_q > 0$ always reduces asset price volatility.

Figure 5: Loss function and volatilities for varying weights on inflation and asset prices.



Note: Unconditional standard deviation of inflation π_t, house prices q_t and output gap $y_t - y_t^{n,RE}$ under the ALM, and loss function $\mathscr{L}$, as a function of ϕ_y and ϕ_q, keeping $\tilde{\omega} = 0.975$ throughout. All values are reported relative to the coefficient ϕ_y reported in Column (3) of Table 1. Red lines denote contour lines at unity, i.e. the value attained by the optimal coefficient ϕ_y with $\phi_q = 0$. Black dots denote the value attained under the unrestricted optimal coefficients, reported in Column (4) of Table 1.

Importantly, the reaction to asset prices does not increase asset price volatility. This is in stark contrast to the rational bubbles of Gali (2014, 2017). Rational bubbles grow at the rate of interest, and so raising rates when a bubble is growing makes it grow even faster, causing more volatility. By contrast, raising rates in the learning model here has the effect of lowering the house price today: A higher real rate requires a higher expected return on housing. For a given expected capital gain $\hat{\mu}_t$, a higher return needs to be brought about by a lower price today. The reduction in the house price today then reduces optimism about future price growth.

7 Extension: General Asset Price Beliefs

One might wonder whether our results hinge in any way on our assumption that agents' subjective beliefs about asset prices are given by a simple random walk with drift. In this section, we show that this is not the case. All our results so far extend to a very general form of beliefs about asset prices that encompass extrapolative as well as attenuating beliefs relative to rational expectations, "natural expectations" (Fuster *et al.*, 2012) and other forms of non-rational beliefs. The only assumptions we have to retain are that expectations are conditionally model-consistent in the sense of Definition 2, and that the subjective law of motion for asset prices is independent of policy. While this second assumption is admittedly somewhat limiting, an environment in which agents do think that monetary policy can curb asset price booms probably provides an even stronger rationale for leaning against the wind that what we discuss here.

We replace the subjective law of motion for asset prices in (3.3)–(3.4) with a general belief of the form:

$$q_t = A(L)\, z_t + B(L)\, u_t.$$

where A and B are arbitrary lag polynomials. Subjective beliefs can depend in an arbitrary way on the fundamental shocks u_t (i.e. productivity and cost-push shocks) as well as a subjective forecast error z_t. The general formula nests rational expectations, our baseline belief system, and a multitude of other forms of subjective beliefs.

Under flexible prices, we can show that the housing demand function in the PLM (i.e. under $\mathscr{P}$), which previously was given by (4.18), is replaced by:

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} + k_q q_t + \tilde{k}_\mu \sum_{s=0}^{\infty} \tilde{\rho}^s \mathbb{E}_t^{\mathscr{P}} \Delta q_{t+s+1}. \tag{7.1}$$

The coefficients k_a, k_h and k_q are the same as in the original model, and moreover we have $\tilde{k}_\mu > 0$ and $0 < \tilde{\rho} < \beta$.The natural real rate under the PLM has a somewhat more convoluted form, but importantly, it is still increasing in expectations of asset price growth:

$$\begin{aligned} r_t^{n,PLM} = r_t^{,RE} &+ \gamma\kappa_1 \left(k_a (2 - \rho_a - k_h)\, a_t - (1-\kappa_h)^2\, h_{t-1}^{n,PLM} + k_q (1-k_h)\, q_t \right) \\ &+ \gamma\kappa_1 \left(\frac{\tilde{k}_\mu}{\tilde{\rho}} - k_q \right) \mathbb{E}_t^{\mathscr{P}} \Delta q_{t+1} + \gamma\kappa_1 \tilde{k}_\mu \left(1 - k_h - \frac{1-\tilde{\rho}}{\tilde{\rho}} \right) \sum_{s=1}^{\infty} \tilde{\rho}^s \mathbb{E}_t^{\mathscr{P}} \Delta q_{t+s+1}. \end{aligned} \tag{7.2}$$

Moreover, since the asset price q_t is independent of policy under the PLM, it drops out of the equations describing the dynamics of the sticky price equilibrium relative to flexible prices. Equations (4.24)–(4.26) continue to hold and the asset price enters only indirectly through the natural real rate $r_t^{n,PLM}$, which itself is independent of policy. As a consequence, all our results from Section 5 continue to hold.

8 Extension: Asset Production

Here, we extend the model to allow for production of the durable asset. In this extended model, asset price misalignments have real effects even under flexible prices because they distort investment decisions. This fundamentally changes the policy tradeoff. It also moves us away from a world in which the effects of non-rational expectations operate only through wealth effects.

Relative to the baseline model, we now assume that the stock of the durable asset depreciates at the rate δ. The representative household owns firms that can produce I_t durable assets from K_t consumption goods. Their production function has decreasing returns to scale:

$$I_t = A_h \cdot K_t^{\omega}. \tag{8.1}$$

Production takes place within one period. The profits of the investment firms are:

$$\Pi_t = Q_t I_t - K_t \tag{8.2}$$

and profit maximization leads to the first order condition:

$$I_t = A_h \left(\omega Q_t A_h\right)^{\frac{\omega}{1-\omega}}. \tag{8.3}$$

The budget constraint of the household becomes

$$C_t + Q_t \left(H_t - (1-\delta) H_{t-1}\right) + \frac{1+i_{t-1}}{1+\pi_t} B_{t-1} \quad = \quad W_t N_t + \Pi_t + T_t + B_t. \tag{8.4}$$

Market clearing in the durable asset market now requires

$$H_t = (1-\delta)\, H_{t-1} + I_t. \tag{8.5}$$

The equilibrium is defined analogously to section 3. Agents do not know the market clearing condition (8.5), but instead hold subjective beliefs that the asset price follows equations (3.3)–(3.4). Beliefs about the hidden state μ_t are updated using the Kalman filter as before, and expectations about the remaining equilibrium objects satisfy conditional model consistency as defined in section 3.

8.1 Linearized equilibrium

We relegate the complete description of the linearized equilibrium to the appendix. Importantly, the natural real rate of interest in the model with asset production continues to be increasing in the asset price belief $\hat{\mu}_t$. As before, we can write the PLM under sticky prices in deviation from the flexible price PLM:

$$\pi_t = \beta \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} + \kappa \left(\hat{c}_t + \kappa_1 \hat{\tilde{h}}_t \right) + \eta_t \tag{8.6}$$

$$i_t = \gamma \left(\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t \right) + \mathbb{E}_t \pi_{t+1} + r_t^{n,PLM} \tag{8.7}$$

$$\theta \hat{h}_t = \gamma \hat{c}_t - \tilde{\beta} \gamma \mathbb{E}_t \hat{c}_{t+1}. \tag{8.8}$$

where $\tilde{h}_t = h_t - (1-\delta) h_t$.

8.2 Welfare functions

We can derive a quadratic approximation of welfare to evaluate different policies. If we evaluate welfare under the actual law of motion, then welfare takes the form $-\sum_{t=0}^{\infty} \mathbb{E}_0 \mathscr{L}_t$, where the period loss function is given by

$$\begin{aligned}
\mathscr{L}_t = {} & 2\sigma \frac{\xi}{1-\xi} \frac{1}{1-\beta\xi} \pi_t^2 + \left(\gamma \frac{\bar{C}}{\bar{Y}} + \frac{1-\alpha+\phi}{\alpha} \frac{\bar{C}^2}{\bar{Y}^2} \right) \left(c_t - c_t^{n,RE} \right)^2 \\
& + \left(\frac{1-\alpha+\phi}{\alpha} \frac{\bar{Q}^2 \bar{H}^2}{\bar{Y}^2} + (1-\omega) \frac{\bar{Q}\bar{H}}{\bar{Y}} \right) \left(\tilde{h}_t - \tilde{h}_t^{n,RE} \right)^2 + 2 \frac{1-\alpha+\phi}{\alpha} \frac{\bar{C}}{\bar{Y}} \frac{\bar{Q}\bar{H}}{\bar{Y}} \left(c_t - c_t^{n,RE} \right) \left(\tilde{h}_t - \tilde{h}_t^{n,RE} \right) \\
& + \left(1 - \beta (1-\delta) \right) \theta \frac{\bar{Q}\bar{H}}{\bar{Y}} \left(h_t - h_t^{n,RE} \right)^2.
\end{aligned} \tag{8.9}$$

Compared to the standard New Keynesian model, we have to take into account variation in the asset stock h_t that the household owns, as well as variations in asset investment $\tilde{h}_t$. As before, the rational expectations equilibrium under flexible prices is first-best efficient, and the loss function can therefore be written in deviations from it.

By contrast, under the PLM, welfare is approximated by $-\sum_{t=0}^{\infty} \mathbb{E}_0^{\mathscr{P}} \mathscr{L}_t^{PLM}$, where the period loss function is given by

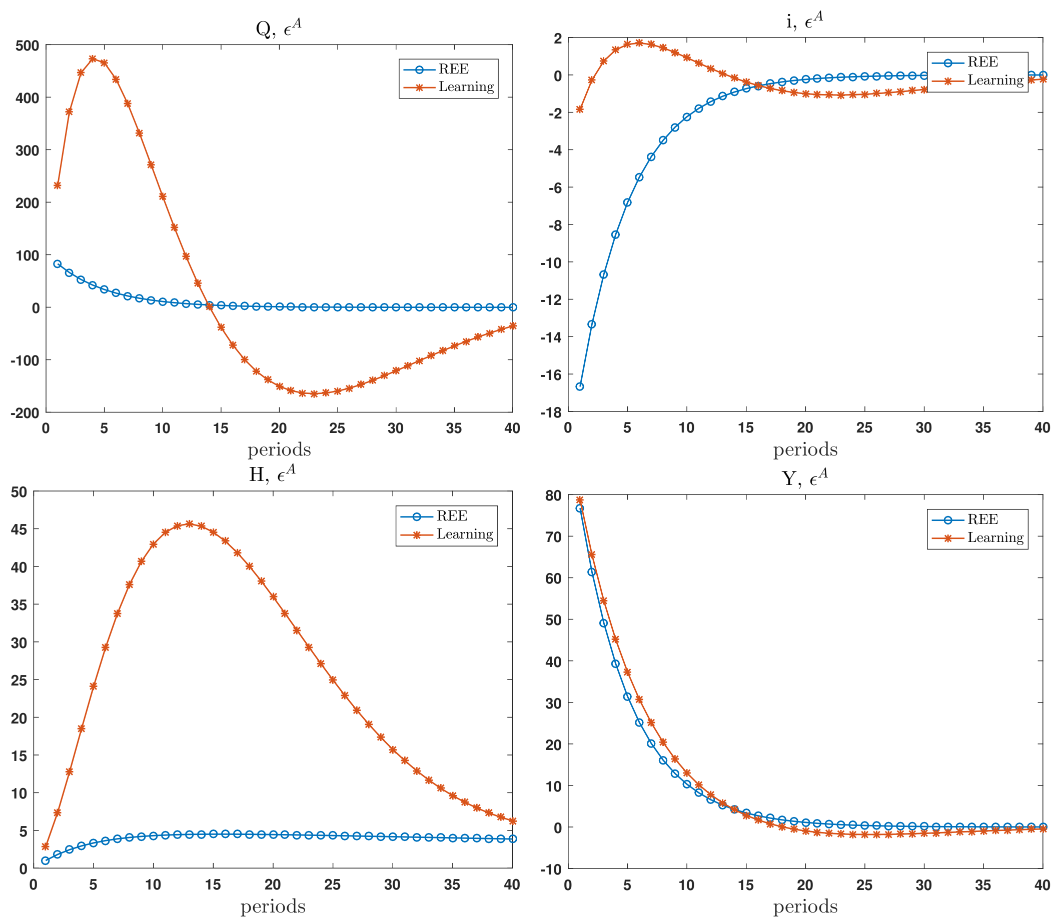
$$\begin{aligned}
\mathscr{L}_t^{PLM} = {} & 2\sigma \frac{\xi}{1-\xi} \frac{1}{1-\beta\xi} \pi_t^2 + \left(\gamma \frac{\bar{Y}}{\bar{C}} + \frac{1-\alpha+\phi}{\alpha} \right) \left(\frac{\bar{C}}{\bar{Y}} \hat{c}_t \right)^2 \\
& + \frac{1-\alpha+\phi}{\alpha} \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} \hat{\tilde{h}}_t \right)^2 + 2 \left(\gamma \frac{\bar{Y}}{\bar{C}} + \frac{1-\alpha+\phi}{\alpha} \right) \left(\frac{\bar{C}}{\bar{Y}} \hat{c}_t \right) \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} \hat{\tilde{h}}_t \right) \\
& - \left(1 - \beta (1-\delta) \right) \theta \frac{\bar{Q}\bar{H}}{\bar{Y}} \hat{h}_t^2.
\end{aligned} \tag{8.10}$$

8.3 Numerical results

We illustrate the dynamics of the model with a simple calibration. We take the same parameters as in the baseline model, and in addition set $\omega = 0.636$, corresponding to an elasticity of asset supply of 1.75. The rate of depreciation of the asset is set to $\delta = 0.007$.

Allowing for variation in the supply of the asset implies that learning introduces and allocative inefficienty into the model. This is illustrated in Figure (6), which compares the responses to a productivity shock under rational expectations and learning. AS in the model with fixed asset supply, learning introduces additional volatility in asset prices. But now, during an asset price boom, asset production responds to the increase in asset demand, diverting resources away from consumption (and leisure).

Figure 6: Effect of learning under flexible prices, with asset production.

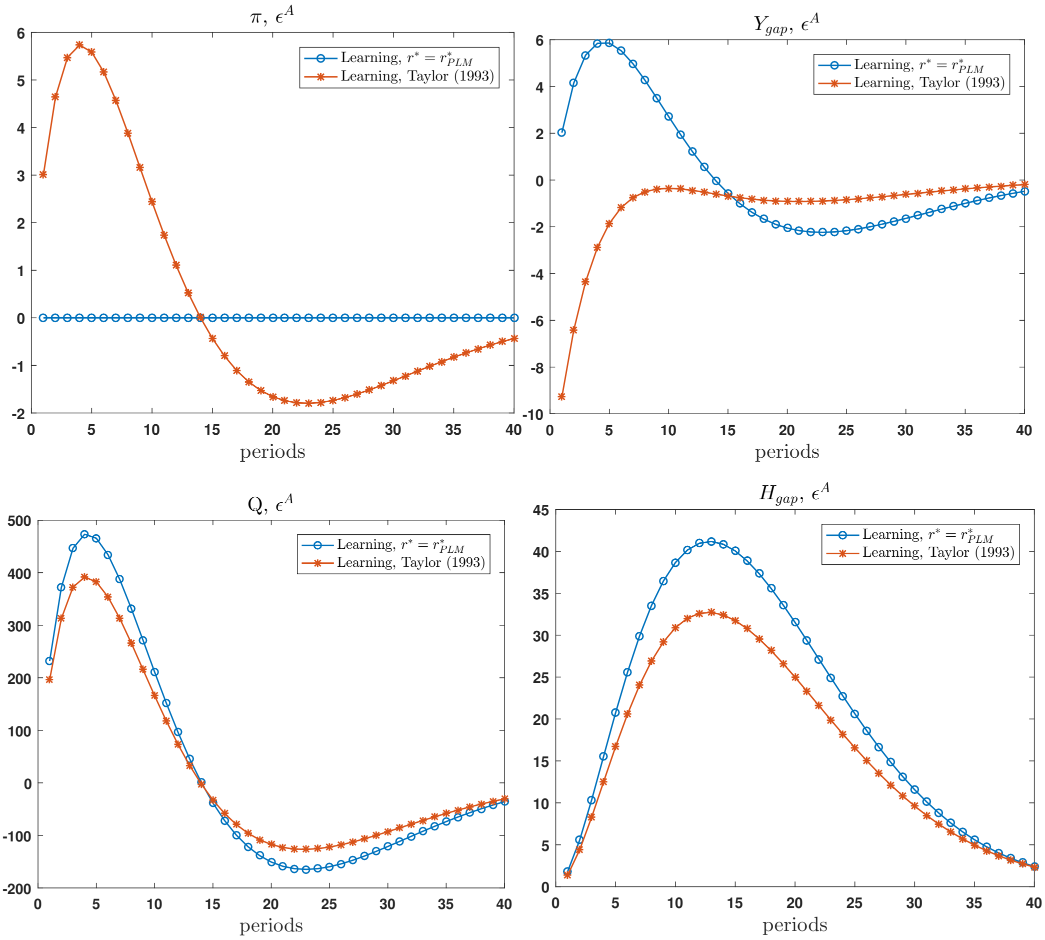


Note: Response to a one standard deviation positive technology shock ε_{At}. Log percentage points. Flexible prices and zero inflation, with asset production.

Nevertheless, the key results outlined in sections 4 and 5 remain intact. As was the case with the baseline model, strict inflation targeting requires the policymaker to track movements in the natural rate of interest in the PLM; but this is no longer optimal.

This is illustrated in figure 7, which shows responses to a one standard deviation productivity shock for two policy rules: The rule(5.3) which follows the PLM natural rate $r_t^{n,PLM}$, and the standard Taylor rule (5.5). The figure shows that the rule following $r_t^{n,PLM}$ achieves strict inflation targeting as before, and therefore implements the flexible price allocation under learning. But this allocation is inefficient because of the investment decisions taken under non-rational expectations. In particular, the response of the housing gap $h_t - h_t^{n,RE}$ is larger under strict inflation targeting than under the Taylor rule, as is the response of asset prices.

Figure 7: Inflation targeting vs Taylor rule, with asset production.



Note: Response to a unit standard deviation positive technology shock ε_{At} under sticky prices and with asset production. Log percentage points. The interest rate rules used are given in Equations (5.3) (using the PLM natural rate with asset production) and (5.5).

As in section 6, we compute optimal simple rules of the form (5.9) for our calibration of the model. The results are tabulated in Table (2). Qualitatively, we obtain the same results as in the model with fixed asset supply, although the optimal coefficients differ.

Table 2: Optimized simple rule coefficients, with asset production.

	RE, $\mathscr{L}$		Learning, $\mathscr{L}$		Learning, $\mathscr{L}^{PLM}$	
	(1)	(2)	(3)	(4)	(5)	(6)
ϕ_y	0.200	0.087	0.0561	0.0518	0.0843	0.0843
ϕ_q		0.755		0.0087		0
ω		0		0		n/a
$\sigma(\pi_t)$	0.323	0.211	0.139	0.106	0.121	0.121
$\sigma\left(c_t - c_t^{n,RE}\right)$	0.459	0.616	0.508	0.537	1.047	1.047
$\sigma\left(h_t - h_t^{n,RE}\right)$	0.140	0.156	1.784	1.804	25.382	25.382
$\sigma(q_t)$	0.928	1.245	1.089	1.150	1.000	1.000
Loss fn value	2.890	1.804	1.114	0.995	1.372	1.372

Note: Loss function values and unconditional standard deviation of inflation π_t, house prices q_t and output gap $y_t - y_t^{n,RE}$ in percentage points. Columns (1) and (2) are evaluated under rational expectations, Columns (3) and (4) under learning.

9 Conclusion

In this paper, we have characterized optimal monetary policy in a model in which agents are learning about asset prices. The model builds on the standard New-Keynesian model with a durable asset in fixed supply. Agents form expectations about asset price in an extrapolative fashion. However, their expectations remain model-consistent conditional on their beliefs about house prices, which allows us to isolate the effects of learning about asset prices from the potentially many other ways in which learning could affect the economy. Learning amplifies asset price fluctuations in the model, and leads to perceived wealth effects that create inefficient fluctuations in aggregate demand.

We have given an analytical solution to the optimal policy in the linearized model with learning, and have showed that flexible inflation targeting remains optimal. However, inflation targeting requires a very different prescription for the nominal interest rate under learning. The interest rate has to increase with asset prices and subjective expectations of future asset price growth. This form of "leaning against the wind" is absent under rational expectations.

We have then showed using numerical simulations that the optimal policy can be reasonably well approximated by simple rules that responds to inflation and a moving average of asset price growth. These rules have the advantage that they can be implemented without knowledge of the state of subjective beliefs. Removing the asset price reaction from the rule seems to cause larger fluctuations particularly in inflation. A reaction to asset prices always reduces asset price volatility, but too much of a reaction can increase inflation volatility. Nonetheless, the optimal weight on asset prices is strictly positive for the rules we considered.

We have also shown that our theoretical results are robust to the specification of subjective beliefs, encompassing a wide range of belief specifications used in the literature. Another extension of our model showed that the case for leaning against the wind is strengthened further when investment decisions are affected by asset prices.

So far, we can only solve for fully optimal policy in the baseline version of our model. Further research should also try to solve for fully optimal monetary policy in our extensions. A particularity in this context is that the policy problem is not easily wrote in a recursive form if the objective function of the policymaker and the first order conditions of the private sector are evaluated under different probability measures. This is not a problem for numerical optimizations but complicates analytical characterizations of optimal policy.

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A Appendix: Details on the derivations

A.1 Asset demand in the model with fixed asset supply

The coefficients in the asset demand function (4.18) are:

$$
\begin{aligned}
k_h &= \frac{1}{2\beta}\left(1+\beta+\frac{\theta}{\gamma\kappa_1}(1-\beta)-\sqrt{\left(1+\beta+\frac{\theta}{\gamma\kappa_1}(1-\beta)\right)^2-4\beta}\right)\in(0,1)\\
k_a &= \frac{1+\phi}{1+\phi-\alpha(1-\gamma)}\frac{1-\beta\rho_a}{\kappa_1+(1-\beta)\frac{\theta}{\gamma}+\beta\kappa_1(1-k_h-\rho_a)}>0\\
k_q &= \frac{1}{\gamma}\frac{1-\beta}{(1-\beta)\frac{\theta}{\gamma}+\kappa_1(1-\beta k_h)}>0\\
k_\mu &= \frac{\beta/\gamma}{(1-\beta)\frac{\theta}{\gamma}+\kappa_1(1-\beta\rho_\mu+\beta(1-k_h))}\frac{(1+\beta)\frac{\theta}{\gamma}+\kappa_1\beta(1-k_h)}{(1+\beta)\frac{\theta}{\gamma}+\kappa_1(1-\beta k_h)}>0
\end{aligned}
$$

A.2 Asset demand with extended beliefs

The coefficients $\tilde{k}_\mu$ and $\tilde{\rho}$ read as follows:

$$
\begin{aligned}
\tilde{k}_\mu &= \frac{1}{\gamma}\frac{\beta}{(1-\beta)\frac{\theta}{\gamma}+\kappa_1(1+\beta-\beta k_h)}\frac{(1+\beta)\frac{\theta}{\gamma}+\kappa_1\beta(1-k_h)}{(1+\beta)\frac{\theta}{\gamma}+\kappa_1(1-\beta k_h)}>0\\
\tilde{\rho} &= \frac{\beta}{\beta+(1-\beta)\frac{\theta}{\gamma\kappa_1}+(1-\beta k_h)}\in(0,1)\,.
\end{aligned}
$$

A.3 Replication of RE-optimal policy with cost push shocks under learning

The solution undear learning must be obtained again by first solving the PLM and then imposing equilibrium. We start by imposing (5.6) as a targeting rule in the PLM. We can make use of the fact that we can represent the equilibrium conditions of the PLM by the three equation (4.24)–(4.26). Substituting the targeting rule yields:

$$
\hat{c}_t = -\frac{1-\zeta(1-\beta\rho)}{\kappa}\eta_t-\kappa_1\left(\hat{h}_t-\hat{h}_{t-1}\right) \tag{A.1}
$$

$$
\hat{h}_t = \frac{\gamma}{\theta(1-\beta)}\left(\hat{c}_t-\beta\mathbb{E}_t^{\mathscr{P}}\hat{c}_{t+1}\right). \tag{A.2}
$$

This system has one state variable, the asset holding gap $\hat{h}_{t-1}$, and one shock, the cost-push shock η_t. Using the method of undetermined coefficients leads to the following solution:

$$\hat{c}_t = a\hat{h}_{t-1} + b\eta_t \tag{A.3}$$

$$\hat{h}_t = \gamma \frac{a\hat{h}_{t-1} + b\left(1-\beta\rho\right)\eta_t}{\theta\left(1-\beta\right)+\gamma\beta a} \tag{A.4}$$

where

$$a = \frac{1-\beta}{\beta}\frac{\sqrt{\left(\theta+\gamma\kappa_1\right)^2 + 4\frac{\beta}{1-\beta}\gamma\kappa_1\theta} - \left(\theta+\gamma\kappa_1\right)}{2\gamma}$$
$$b = -\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\left(1+\frac{\kappa_1\gamma\left(1-\beta\rho\right)}{\theta\left(1-\beta\right)+\gamma\beta a}\right)^{-1}.$$

We can substitute this solution into the IS equation (4.25) to solve for the implied nominal interest rate path to obtain:

$$i_t = r_t^{n,PLM} + \rho\eta_t - \gamma\left(a\left(\hat{h}_t - \hat{h}_{t-1}\right) + b\left(1-\rho\right)\eta_t\right).$$

We can equivalently rewrite the purchases of assets in terms of the perceived output gap and inflation by observing that $\eta_t = \pi_t/\zeta$ and

$$\begin{aligned}
\kappa_1\left(\hat{h}_t - \hat{h}_{t-1}\right) &= -\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\eta_t - \hat{c}_t \\
&= -\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\eta_t - \hat{y}_t + \frac{QH}{Y}\left(\hat{h}_t - \hat{h}_{t-1}\right) \\
\Leftrightarrow \hat{h}_t - \hat{h}_{t-1} &= \frac{1+\phi-\alpha}{\kappa_1\alpha\gamma}\left(\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\eta_t + \hat{y}_t\right).
\end{aligned}$$

We can then express the interest rate as a function of the perceived natural rate, inflation and the perceived output gap:

$$i_t = r_t^{n,PLM} + a\frac{1+\phi-\alpha}{\kappa_1\alpha}\hat{y}_t + \left(\rho - \gamma\left(1-\rho\right)b + a\frac{1+\phi-\alpha}{\kappa_1\alpha}\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\right)\frac{1}{\zeta}\pi_t. \tag{A.5}$$

This expression shows that the equilibrium interest rate path depends on inflation and the perceived output gap as well as the natural rate. The natural rate is increasing in the level of asset prices as well as the subjective expectation of future asset price growth. Therefore, the nominal interest rate in (A.5) is effectively leaning against the wind.

In the second step, we solve for the equilibrium (ALM) allocation and prices. Imposing asset market clearing requires $h_t = 0$, which can be rewritten as

$$h_t^{n,PLM} = -\hat{h}_t. \tag{A.6}$$

Substituting this into the expression for asset demand (4.18), we can solve for the equilibrium price:

$$\begin{aligned} -\hat{h}_t &= k_a a_t - k_h \hat{h}_{t-1} + k_q q_t + k_\mu \hat{\mu}_{t-1} \\ \Rightarrow q_t &= -\frac{1}{k_q}\left(k_a a_t + k_\mu \hat{\mu}_{t-1} + \hat{h}_t - k_h \hat{h}_{t-1}\right). \end{aligned} \tag{A.7}$$

Finally, we solve for the equilibrium output gap. We first rewrite Equation (4.19) in terms of housing purchases under flexible prices:

$$y_t^{n,PLM} = y_t^{n,RE} + \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha}\left(h_t^{n,PLM} - h_{t-1}^{n,PLM}\right)$$

and use this relationship to arrive at the main result in this section:

$$\begin{aligned} y_t - y_t^{n,RE} &= \hat{y}_t + y_t^{n,PLM} - y_t^{n,RE} \\ &= \hat{c}_t + \frac{\bar{Q}\bar{H}}{\bar{Y}}\left(\hat{h}_t - \hat{h}_{t-1}\right) + y_t^{n,PLM} - y_t^{n,RE} \\ &= \hat{c}_t + \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha}\right)\left(\hat{h}_t - \hat{h}_{t-1}\right) \\ &= \hat{c}_t + \kappa_1\left(\hat{h}_t - \hat{h}_{t-1}\right) \\ &= -\frac{1-\zeta\left(1-\beta\rho\right)}{\kappa}\eta_t. \end{aligned}$$

That is, the equilibrium path of the output gap is identical to that under rational expectations.

A.4 Linearized equilibrium conditions and natural rate in the model with asset production

Under rational expectations, the following set of equations describe the linearized equilibrium (up to a monetary policy rule):

$$y_t = a_t + \alpha n_t \tag{A.8}$$

$$\pi_t = \beta \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi}\left(w_t - a_t + (1-\alpha)\, n_t\right) + \eta_t \tag{A.9}$$

$$w_t = \gamma c_t + \phi n_t \tag{A.10}$$

$$\bar{Y} y_t = \bar{C} c_t + \bar{Q}\bar{H}\left(h_t - (1-\delta)\, h_{t-1}\right) \tag{A.11}$$

$$h_t = (1-\delta)\, h_{t-1} + \frac{\omega\delta}{1-\omega} q_t \tag{A.12}$$

$$i_t = \gamma\left(\mathbb{E}_t c_{t+1} - c_t\right) + \mathbb{E}_t \pi_{t+1} \tag{A.13}$$

$$q_t = \gamma c_t - \left(1 - \beta(1-\delta)\right)\theta h_t - \beta(1-\delta)\gamma \mathbb{E}_t c_{t+1} + \beta\delta \mathbb{E}_t q_{t+1}. \tag{A.14}$$

We can eliminate the asset price, wages, labor and output from this to get a three-equation system:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi}\left(\gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha}\right)\left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}}\tilde{h}_t\right) - \frac{1+\phi}{\alpha} a_t\right) + \eta_t \tag{A.15}$$

$$i_t = \gamma\left(\mathbb{E}_t c_{t+1} - c_t\right) + \mathbb{E}_t \pi_{t+1} \tag{A.16}$$

$$\frac{1-\omega}{\omega\delta}\tilde{h}_t = \gamma c_t - \left(1-\tilde{\beta}\right)\theta h_t - \tilde{\beta}\gamma \mathbb{E}_t c_{t+1} + \tilde{\beta}\frac{1-\omega}{\omega\delta}\mathbb{E}_t \tilde{h}_{t+1}. \tag{A.17}$$

where we have written $\tilde{h}_t = h_t - (1-\delta)\, h_{t-1}$ and $\tilde{\beta} = (1-\delta)\,\beta$ to ease the notation. The equations are a bit long and ugly but that's what it is for now.

Under learning, we can do a similar exercise. We first tackle the PLM. All we do is to replace the market clearing condition with the subjective law of motion for asset prices:

$$y_t = a_t + \alpha n_t \tag{A.18}$$

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi}\left(w_t - a_t + (1-\alpha)\, n_t\right) + \eta_t \tag{A.19}$$

$$w_t = \gamma c_t + \phi n_t \tag{A.20}$$

$$\bar{Y} y_t = \bar{C} c_t + \bar{Q}\bar{H}\left(h_t - (1-\delta)\, h_{t-1}\right) \tag{A.21}$$

$$i_t = \gamma\left(\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t\right) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \tag{A.22}$$

$$q_t = \gamma c_t - \left(1 - \beta(1-\delta)\right)\theta h_t - \beta(1-\delta)\gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta\delta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \tag{A.23}$$

$$q_t = q_{t-1} + \hat{\mu}_{t-1} + z_t \tag{A.24}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t. \tag{A.25}$$

The three-equation system describing the equilibrium boils down to:

$$\pi_t = \beta \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} + \frac{(1-\xi)\left(1-\beta\xi\right)}{\xi}\left(\gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha}\right)\left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}}\tilde{h}_t\right) - \frac{1+\phi}{\alpha} a_t\right) + \eta_t \tag{A.26}$$

$$i_t = \gamma\left(\mathbb{E}_t^{\mathscr{P}} c_{t+1} - c_t\right) + \mathbb{E}_t^{\mathscr{P}} \pi_{t+1} \tag{A.27}$$

$$q_t = \gamma c_t - \left(1-\tilde{\beta}\right)\theta h_t - \tilde{\beta}\gamma \mathbb{E}_t^{\mathscr{P}} c_{t+1} + \tilde{\beta}\mathbb{E}_t^{\mathscr{P}} q_{t+1}. \tag{A.28}$$

The flexible price equilibrium under rational expectations is characterized as:

$$h_t^{n,RE} = k_a^{RE} a_t + k_h^{RE} h_{t-1}^{RE},\ k_a^{RE} > 0,\ k_h^{RE} \in (0, 1-\delta) \tag{A.29}$$

$$y_t^{n,RE} = \left(\frac{\bar{C}}{\bar{Y}} \frac{\phi+1}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma} + \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\bar{C}}{\bar{Y}}\kappa_1\right) k_a^{RE}\right) a_t - \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\bar{C}}{\bar{Y}}\kappa_1\right)\left(1-\delta-k_h^{RE}\right) h_{t-1}^{RE} \tag{A.30}$$

$$c_t^{n,RE} = \left(\frac{\phi+1}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma} - \kappa_1 k_a^{RE}\right) a_t + \kappa_1\left(1-\delta-k_h^{RE}\right) h_{t-1}^{RE} \tag{A.31}$$

$$r_t^{n,RE} = -\gamma\left(\frac{\phi+1}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma} - \kappa_1 k_a^{RE} + \kappa_1\left(1-\delta-k_h^{RE}\right) k_a^{RE}\right)\left(1-\rho_a\right) a_t - \gamma\kappa_1\left(1-\delta-k_h^{RE}\right)\left(1-k_h^{RE}\right) h_{t-1}^{RE} \tag{A.32}$$

where the constants κ and κ_1 are now redefined as:

$$\kappa = \frac{1-\xi}{\xi}\frac{1-\beta\xi}{\alpha}\left(\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma\right)$$
$$\kappa_1 = \frac{1+\phi-\alpha}{\frac{\bar{C}}{\bar{Y}}\left(1+\phi-\alpha\right)+\alpha\gamma}\frac{\bar{Q}\bar{H}}{\bar{Y}}.$$

Under learning and flexible prices, we can boil things down to these two equations to solve for the PLM:

$$0 = \gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha}\right)\left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}}\tilde{h}_t\right) - \left(1+\phi\right) a_t \tag{A.33}$$

$$q_t = \gamma c_t - \left(1-\beta\left(1-\delta\right)\right)\theta h_t - \beta\left(1-\delta\right)\gamma \mathbb{E}_t c_{t+1} + \beta\delta\mathbb{E}_t^{\mathscr{P}} q_{t+1}. \tag{A.34}$$

Guess and verify

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} + k_q q_t + k_\mu \hat{\mu}_{t-1} \tag{A.35}$$

where the coefficients are given by:

$$k_h = \frac{1}{2\tilde{\beta}}\left(1+\tilde{\beta}(1-\delta)+\frac{\theta}{\gamma\kappa_1}\left(1-\tilde{\beta}\right)-\sqrt{\left(1+\tilde{\beta}(1-\delta)+\frac{\theta}{\gamma\kappa_1}\left(1-\tilde{\beta}\right)\right)^2-4\tilde{\beta}(1-\delta)}\right)\in(0,1-\delta)$$

$$k_a = \frac{1+\phi}{\frac{\bar{C}}{\bar{Y}}(1+\phi-\alpha)+\alpha\gamma}\frac{1-\tilde{\beta}\rho_a}{\kappa_1+\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma}+\tilde{\beta}\kappa_1\left(1-\delta-k_h-\rho_a\right)}>0$$

$$k_q = -\frac{1}{\gamma}\frac{1-\tilde{\beta}}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma}+\kappa_1\left(1-\tilde{\beta}k_h-\tilde{\beta}\delta\right)}<0$$

$$k_\mu = \frac{\tilde{\beta}/\gamma}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma}+\kappa_1\left(1-\tilde{\beta}\rho_\mu+\tilde{\beta}(1-\delta-k_h)\right)}\frac{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma}+\kappa_1\tilde{\beta}(1-\delta-k_h)}{\left(1-\tilde{\beta}\right)\frac{\theta}{\gamma}+\kappa_1\left(1-\tilde{\beta}k_h-\tilde{\beta}\delta\right)}>0$$

We can characterize the flex-price PLM consumption, output and interest rates:

$$\begin{aligned}
\frac{c_t^{n,PLM}-c_t^{n,RE}}{\kappa_1} &= -\left(k_a-k_a^{RE}\right)a_t-\left(k_h-k_h^{RE}\right)h_{t-1}+\left(1-\delta-k_h^{RE}\right)\left(h_{t-1}^{n,RE}-h_{t-1}^{n,RE}\right)\\
&\quad -k_q q_t-k_\mu\hat{\mu}_{t-1} && \text{(A.36)}\\
\frac{y_t^{n,PLM}-y_t^{n,RE}}{\frac{\bar{Q}\bar{H}}{\bar{Y}}-\frac{\bar{C}}{\bar{Y}}\kappa_1} &= -\frac{c_t^{n,PLM}-c_t^{n,RE}}{\kappa_1} && \text{(A.37)}\\
-\frac{r_t^{n,PLM}-r_t^{n,RE}}{\gamma\kappa_1} &= -\left((1-k_h)(1-\delta-k_h)-\left(1-k_h^{RE}\right)\left(1-\delta-k_h^{RE}\right)\right)h_{t-1}\\
&\quad -\left(1-k_h^{RE}\right)\left(1-\delta-k_h^{RE}\right)\left(h_{t-1}^{n,RE}-h_{t-1}^{n,RE}\right)\\
&\quad +\left(k_a\left(2-\delta-k_a-\rho_a\right)-k_a^{RE}\left(2-\delta-k_a^{RE}-\rho_a\right)\right)a_t\\
&\quad +\left(k_\mu\left(2-\delta-k_h-\rho_\mu\right)-k_q\right)\hat{\mu}_{t-1}+k_q(1-\delta-k_h)q_t && \text{(A.38)}
\end{aligned}$$

In order to find the ALM under flexible prices, we impose market clearing for the durable asset and obtain:

$$\begin{aligned}
(1-\delta)h_{t-1}+\frac{\omega\delta}{1-\omega}q_t &= k_a a_t+k_h h_{t-1}+k_q q_t+k_\mu\hat{\mu}_{t-1}\\
\Leftrightarrow q_t &= \frac{1}{\frac{\omega}{1-\omega}\delta-k_q}\left(k_a a_t+k_\mu\hat{\mu}_{t-1}-(1-\delta-k_h)h_{t-1}\right). && \text{(A.39)}
\end{aligned}$$

The equilibrium price is increasing in productivity, increasing in asset price beliefs, and decreasing in the existing stock of housing.

When the equilibrium asset price q_t from equation (A.39) is substituted into the expression for the natural rate $r_t^{n,PLM}$, the natural real rate in the ALM $r_t^{n,ALM}$ is increasing in asset price expectations $\hat{\mu}_{t-1}$, just as in the baseline model. The sign of the other coefficients are ambiguous and depend on the parameterization.