Topology and Material Optimization via Mathematical Programming

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June 2018

Structural optimization

The goal is to improve behavior of a mechanical structure while keeping its structural properties.

Objectives/constraints:
weight, stiffness, stress, vibration modes, stability

Control variables:
shape $\rightarrow$ shape optimization
material properties $\rightarrow$ topology/material optimization

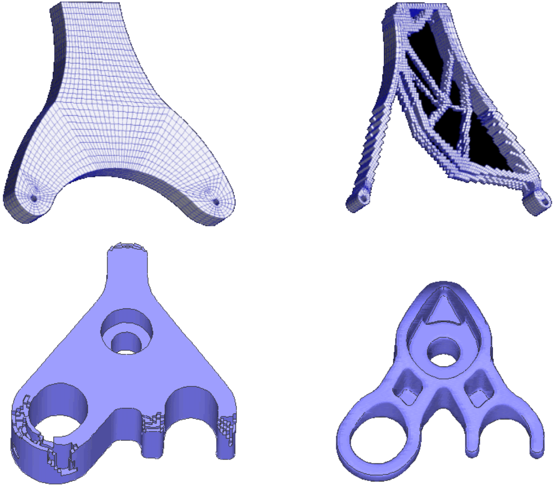
Topology optimization

The goal is to improve behavior of a mechanical structure while keeping its structural properties.

Objectives/constraints:
weight, stiffness, vibration modes, stability, stress

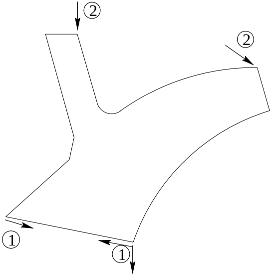
Control variables:
thickness/density (topology optimization, TO)
material properties (FMO)

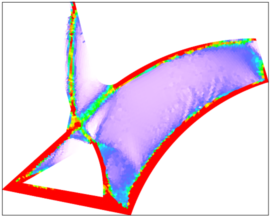
Topology optimization

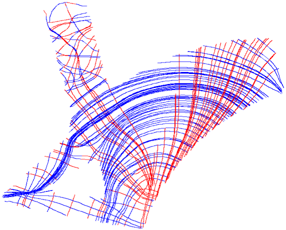


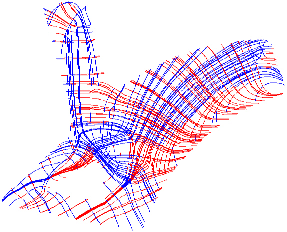
Images courtesy of FE-Design and BMW Motoren GmbH

FMO



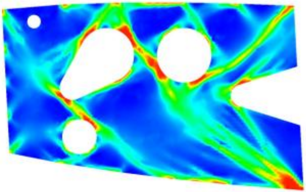


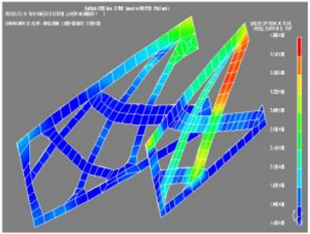




FMO

FMO





Static equilibrium

Weak formulation:
Find $u \in U := \{u \in \mathbb{H} | u_{\Gamma_0} = 0\}$ such that

$$\underbrace{\int_\Omega e(u)(x)^\top \cdot E \cdot e(v)(x)\mathrm{dx}}_{a_E(u,v)} = \underbrace{\int_\Gamma f(x)^\top v(x)\mathrm{dx}}_{l(v)} \quad \forall v \in U.$$

strain energy   work

Static equilibrium

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$$E = \begin{pmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ & E_{2222} & \sqrt{2}E_{2212} \\ \text{sym.} & & 2E_{1212} \end{pmatrix}$$

$$E \in \mathbb{S}^6 \quad \text{in 3D case}$$

Static equilibrium

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$E(x) = \rho(x) E_0$ with $0 \le \rho(x) \le 1$ ... topology optimization

$E(x) = \rho(x)^p E_0$ with $0 \le \rho(x) \le 1$ $\quad (p = 3)$ ... SIMP

E_0 a given (homogeneous, isotropic) material

Static equilibrium

Weak formulation:
Find $u \in U := \{u \in \mathbb{H} | u_{\Gamma_0} = 0\}$ such that

$$\underbrace{\int_\Omega e(u)(x)^\top \cdot E \cdot e(v)(x) \mathrm{dx}}_{a_E(u,v)} = \underbrace{\int_\Gamma f(x)^\top v(x) \mathrm{dx}}_{l(v)} \quad \forall v \in U.$$

$E(x) \in (L^\infty)^{6\times 6}$ . . . free material optimization

Aim:

Given an amount of material, boundary conditions and external load f, find the material (distribution) so that the body is as stiff as possible under f.

Equilibrium

Weak formulation:
Find $u \in U := \{u \in \mathbb{H} | u_{\Gamma_0} = 0\}$ such that

$$\underbrace{\int_\Omega e(u)(x)^\top \cdot E \cdot e(v)(x)\mathrm{dx}}_{a_E(u,v)} = \underbrace{\int_\Gamma f(x)^\top v(x)\mathrm{dx}}_{l(v)} \quad \forall v \in U.$$

strain energy work

Discretization:

$$K(E)u = f, \qquad K(E) = \sum_{i=1}^{m} \sum_{j=1}^{G} B_{i,j} E_i B_{i,j}^\top$$

$E = \rho E_0$ (VTS), $E = \rho^p E_0$ (SIMP), $E \in (L^\infty)^{6\times 6}$ (FMO)

TO primal problem

$$\min_{u,\ldots,u^L,\,\rho} \sum_{i=1}^{m} \rho_i$$

subject to

$$\underline{\rho} \le \rho_i \le \overline{\rho} \quad i = 1, \ldots, m$$

$$(f^\ell)^T u^\ell \le \gamma, \quad \ell = 1, \ldots, L$$

$$K(\rho) u^\ell = f^\ell, \quad \ell = 1, \ldots, L$$

- nonconvex nonlinear programming problem.

TO, reduced primal problem

$$\min_{\rho} \sum_{i=1}^{m} \rho_i$$

subject to

$$\underline{\rho} \leq \rho_i \leq \overline{\rho}, \quad i = 1, \ldots, m$$

$$(f^\ell)^T K(\rho)^{-1} f^\ell \leq \gamma, \quad \ell = 1, \ldots, L$$

- convex nonlinear programming problem
- complexity grows linearly with L

TO, linear SDP primal problem

$$\min_{\rho} \sum_{i=1}^{m} \rho_i$$

subject to

$$\underline{\rho} \leq \rho_i \leq \overline{\rho}, \quad i = 1, \ldots, m$$

$$\begin{pmatrix} \gamma & (f^\ell)^T \\ f^\ell & K(\rho) \end{pmatrix} \succeq 0, \quad \ell = 1, \ldots, L$$

- linear semidefinite programming problem
- L (very) large and sparse SDP constraints

FMO primal problem

$$\min_{u,\ldots,u^L,\, E} \sum_{i=1}^{m} \mathrm{Tr}(E_i)$$

subject to

$$E_i \succeq 0, \quad i = 1, \ldots, m$$
$$\underline{\rho} \leq \mathrm{Tr}(E_i) \leq \overline{\rho} \quad i = 1, \ldots, m$$
$$(f^\ell)^T u^\ell \leq \gamma, \quad \ell = 1, \ldots, L$$
$$K(E) u^\ell = f^\ell, \quad \ell = 1, \ldots, L$$

- nonlinear nonconvex semidefinite programming problem.

FMO, reduced primal problem

$$\min_{E} \sum_{i=1}^{m} \mathrm{Tr}(E_i)$$

subject to

$$E_i \succeq 0, \quad i = 1, \ldots, m$$

$$\underline{\rho} \leq \mathrm{Tr}(E_i) \leq \overline{\rho}, \quad i = 1, \ldots, m$$

$$(f^\ell)^T K(E)^{-1} f^\ell \leq \gamma, \quad \ell = 1, \ldots, L$$

- nonlinear convex semidefinite programming problem
- complexity grows linearly with L.

FMO, linear SDP primal problem

$$\min_E \sum_{i=1}^m \mathrm{Tr}(E_i)$$

subject to

$$E_i \succeq 0, \quad i = 1, \ldots, m$$

$$\underline{\rho} \le \mathrm{Tr}(E_i) \le \overline{\rho}, \quad i = 1, \ldots, m$$

$$\begin{pmatrix} \gamma & (f^\ell)^T \\ f^\ell & K(E) \end{pmatrix} \succeq 0, \quad \ell = 1, \ldots, L$$

- linear semidefinite programming problem
- L very large and sparse SDP constraints

Summary—TO/FMO primal models

There are two classes of models, one based on the primal and one on the dual formulation of the problem.

Primal formulations

- difficult optimisation problems:
  - nonconvex semidefinite programming (SDP) problem
  - convex nonlinear SDP problem
  - large scale linear SDP problem
- N-SDP does not satisfy the Mangasarian-Fromowitz constraint qualification.

FMO models: additional constraints

So far we considered the “basic” topology optimization problem.

Optimal topology/material can change significantly when we add some important[1] constraints.

- Vibration (self-vibration modes)
- Stability w.r.t. buckling
- Displacement constraints
- Stress constraints

The resulting optimization problem can become much more complicated.

[1] Importance depends on the application!

Vibration constraints

The fundamental frequency of the optimal structure is bigger than or equal to a given frequency.

Self-vibrations of the (discretized) structure—eigenvalues of

$$K(E)w = \lambda M(E)w$$

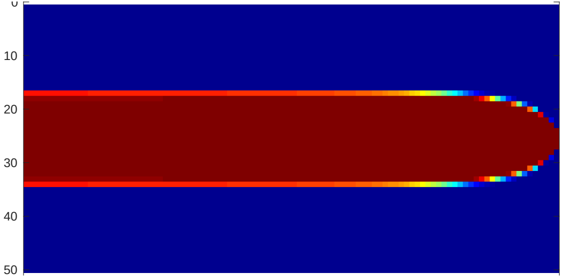
where the mass matrix $M(E)$ has the same sparsity as $K(E)$.

Low vibrations dangerous $\rightarrow$ constraint $\lambda_{\min} \geq \hat{\lambda}$

Equivalently: $K(E) - \hat{\lambda} M(E) \succeq 0$

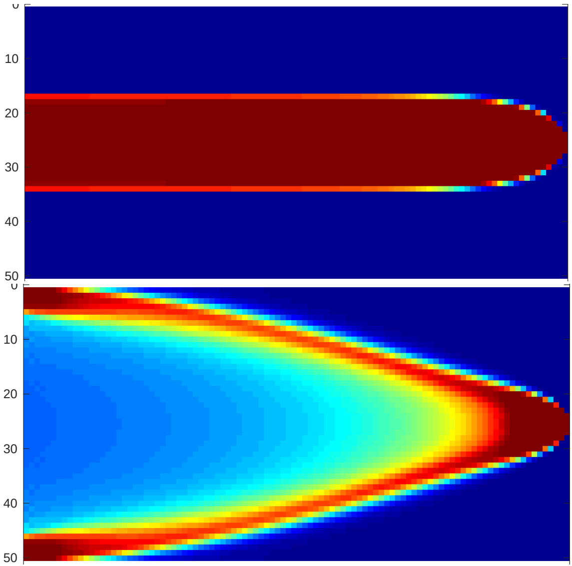
Large-scale SDP constraints $\rightarrow$ use SDP formulation of TO/FMO

Example—vibration constraint



TO without vibration constraint

Example—vibration constraint



TO without and with vibration constraint

Global stability (buckling) constraints

The GEVP

$$K(E)w = \lambda G(E,u)w,$$

where $G(E,u)$ is the geometry stiffness matrix of the structure (depending nonlinearly on E and displacement u).

Buckling constraint:

$$\lambda(E,u) \notin (0,1)$$

Buckling constraints in primal FMO

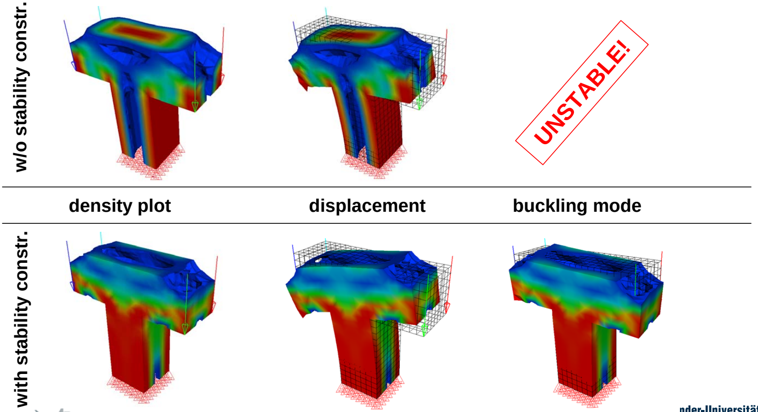
$$K(E) + G(E,u) \succeq 0\,.$$

Buckling constraints in reduced primal FMO

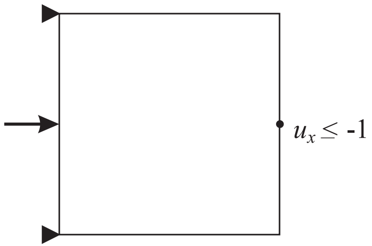
$$K(E) + G(E, K^{-1}(E)f) \succeq 0\,.$$

Case studies: Tc12

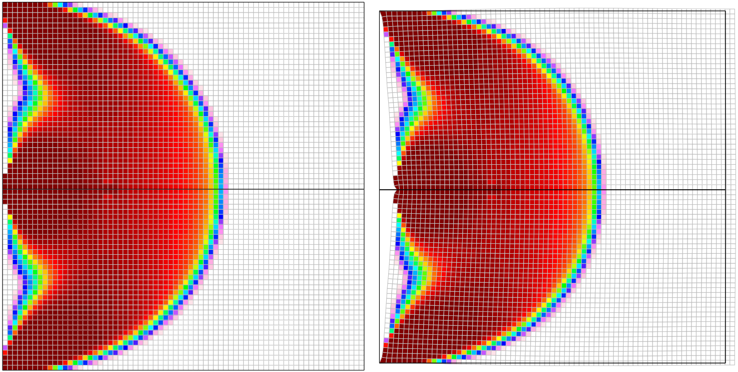
50.000 design variables (sizing), 4 LC + global stability constraints



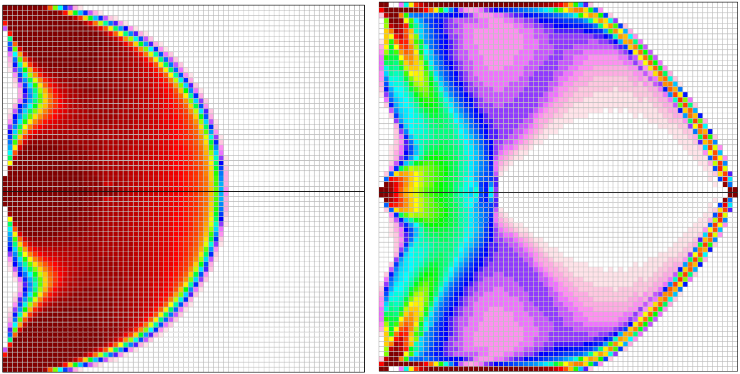
Displacement constraints: actuator design



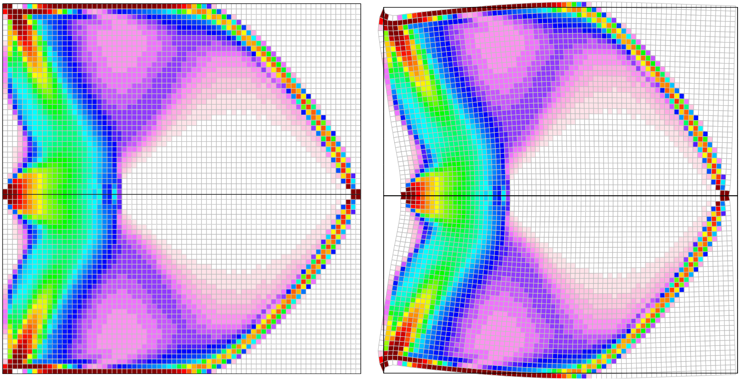
Displacement constraints: actuator design



Displacement constraints: actuator design



Displacement constraints: actuator design



Displacement constraints: actuator design

Stress constraints

Continuous formulation:

restrict norm $\|\sigma(x)\|_{vM}$ for all $x \in \Omega$, $\quad \sigma(x) = Ee(u(x))$

Finite element approximation:

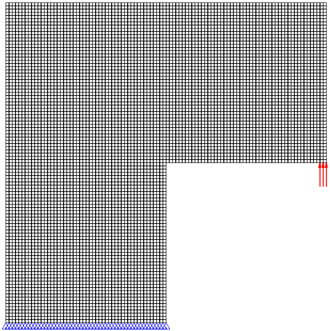
we use integral form of stress constraints

$$\int_{\Omega_i} \|\sigma\|_{vM}^2 \leq s_\sigma |\Omega_i|\,;$$

The von Mises (semi)norm $\|\cdot\|_{vM}$ defined as

$$\|\sigma\|_{vM}^2 := \sigma^\top M\sigma, \quad \text{with } M = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Example: L-shape domain



Example: L-shape domain, TO

For the TO problem, the only way to remove the stress singularity is to change the geometry of the domain, to replace the sharp corner by a smooth arc.

Example: L-shape domain, TO

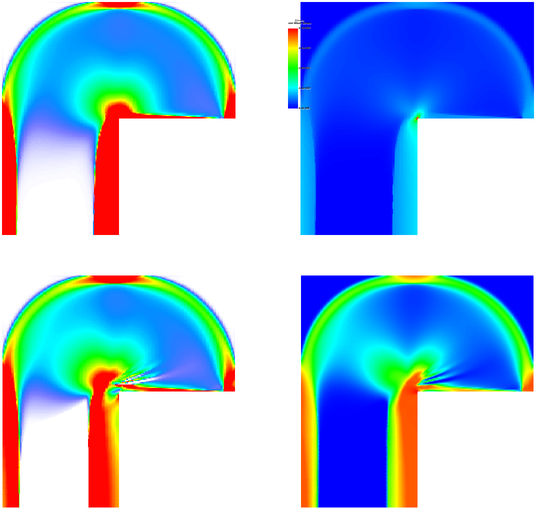


Figure: Problem TC04-s4, TO, without and with stress constraints.