# A class of generalized potential games and applications

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# Optimization problem

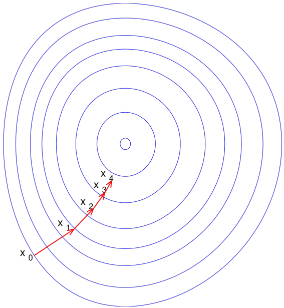
- Traditional optimization problem: minimization of a *single* objective function over some admissible set $A$:

$$\min_{x \in A} f(x).$$

- The standard gradient descent method[1]

1. start with a guess $x_0$,
2. follow the direction of the negative of gradient of $f$:

$$x_{n+1} = x_n - \gamma_n \nabla f(x_n).$$




[1] picture: https://en.wikipedia.org/wiki/Gradient_descent

# Multi-player differentiable games

$n$-player differentiable games:

- players $[n] = \{1, \ldots, n\}$,
- controllable variables: $D_1 \times \ldots \times D_n$ where $D_i \subset \mathbb{R}^{d_i}$ and $\sum_{i=1}^{n} d_i = d$.
- twice continuously differentiable loss functions:

$$\begin{aligned} \ell_i : D_1 \times \ldots \times D_n &\longrightarrow \mathbb{R} \\ \mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) &\mapsto \ell_i(\mathbf{w}_1, \ldots, \mathbf{w}_n). \end{aligned}$$

The objective of each player is to minimize its loss function:

$$\mathbf{w}_i^* = \mathrm{argmin}_{\mathbf{w}_i \in D_i} \ell_i(\mathbf{w}).$$

$\mathbf{w}^*$-Nash equilibrium:

$$\mathbf{w}_i^* = \mathrm{argmin}_{\mathbf{w}_i \in D_i} \ell_i(\mathbf{w}_i, \mathbf{w}_{-i}^*) \quad \forall i.$$

# Potential games, Monderer-Shapley 1996

Weighted potential game: there is a function $\phi : \mathbb{R}^d \to \mathbb{R}$ and positive numbers (weights) $\{\alpha_i\}_{i=1}^n$ such that

$$\alpha_i \nabla_{\mathbf{w}_i} \ell_i = \nabla_{\mathbf{w}_i} \phi \qquad \forall i = 1, \ldots, n.$$

when $\alpha_i = 1 \quad \forall i$: exact potential game.

**Theorem (Monderer-Shapley 1996)**

- *equivalent characterization:*

$$\alpha_i \nabla^2_{\mathbf{w}_i \mathbf{w}_j} \ell_i = \alpha_j \nabla^2_{\mathbf{w}_i \mathbf{w}_j} \ell_j \qquad \forall\, i, j.$$

- *potential function (uniquely defined up to an additive constant): for any $z \in C([0,1], \mathbb{R}^d)$ : $\quad z\big|_{t=0,1} = z_{0,1}$:*

$$\phi(z_1) - \phi(z_0) = \sum_{i=1}^n \alpha_i \int_0^1 z_i'(t) \cdot \nabla_{\mathbf{w}_i} \ell_i(z(t))\, dt.$$

# Potential games

Why potential games?: minimization of the single potential function instead of $n$ loss functions. Existence and attainability of Nash equilibria

Many applications: economics, theoretical computer science, computational social science and sociology, wireless networks, and recently in machine learning

Lloyd S. Shapley (1923-2016): Nobel prize in Economics 2012.

# Can we enlarge the class of potential games?

The simultaneous gradient of the game:

$$\xi(\mathbf{w}) = (\nabla_{\mathbf{w}_1}\ell_1(\mathbf{w}), \ldots, \nabla_{\mathbf{w}_n}\ell_n(\mathbf{w}))^T.$$

weighted potential games:

$$\xi(\mathbf{w}) = M^{-1}\nabla\phi(\mathbf{w}) \quad \text{where} \quad M = \mathrm{dia}(\alpha_1, \ldots, \alpha_n).$$

*linear relation between $\xi$ and $\nabla\phi$*

Our aim is to extend the class of potential games allowing nonlinear relation between $\xi$ and $\nabla\phi$ using a dissipation potential.

# Dissipation potential and its dual

dissipation potential: $\Psi : \mathbb{Z} \times T\mathbb{Z} \to \mathbb{R}$ is called a dissipation potential if for all $z \in \mathbb{Z}$

1. $\Psi(z, \cdot)$ is convex in the second argument,
2. $\min \Psi(z, \cdot) = 0$ and
3. $\Psi(z, 0) = 0$.

Legendre-Fenchel dual (convex dual):

$$\Psi^*(z, \zeta) = \sup_{s \in T_z\mathbb{Z}} \big\{ \langle \zeta, s \rangle - \Psi(z, s) \big\} \quad \text{for} \quad \zeta \in T_z^*\mathbb{Z}.$$

equivalent statements, Fenchel 1949:

$$(i)\, \xi \in \mathbf{D}\Psi(s) \quad \Longleftrightarrow \quad (ii)\, s \in \mathbf{D}\Psi^*(\xi) \quad \Longleftrightarrow \quad (iii)\, \Psi(s) + \Psi^*(\xi) = \langle \xi, s \rangle$$

# Our work: generalized potential games

Generalized potential games: if there exists a dissipation potential $\Psi$ and a potential function $\phi$ such that

$$\xi(\mathbf{w}) = \mathbf{D}_\zeta \Psi^*(\nabla_{\mathbf{w}} \phi(\mathbf{w})).$$

$\Psi(s) = \frac{1}{2} s^T M s$: recover weighted potential games (exact, $M = I$).

## Theorem

- *generalized potential* $\iff \mathbf{D}^2\Psi(\xi(\mathbf{w}))\mathbf{H}(\mathbf{w})$ *is symmetric* ($\mathbf{H}(\mathbf{w})$*: Hessian matrix,* $[\mathbf{H}(\mathbf{w})]_{ij} = \nabla^2_{ij}\ell_i$*).*
- *for all* $z \in C([0,1], \mathbb{R}^d), \quad z\big|_{t=0,1} = z_{0,1}$*:*

$$\phi(z_1) - \phi(z_0) = \int_0^1 z'(t) \cdot \mathbf{D}_s\Psi(\xi(z(t)))\, dt$$

# Generalized potential games: example

Consider a $n$-player game with

$$\ell_1(\mathbf{w}) = \sum_{r=1}^{R} \Big( \frac{k^r_{\mathrm{fw}}}{\alpha^r_1 + 1} w_1^{\alpha^r_1+1} w_2^{\alpha^r_2} \cdots w_n^{\alpha^r_n} - \frac{k^r_{\mathrm{bw}}}{\beta^r_1 + 1} w_1^{\beta^r_1+1} w_2^{\alpha^r_2} \cdots w_n^{\beta^r_n} \Big) \Big( \alpha^r_1 - \beta^r_1 \Big)$$

$$\vdots$$

$$\ell_n(\mathbf{w}) = \sum_{r=1}^{R} \Big( \frac{k^r_{\mathrm{fw}}}{\alpha^r_n + 1} w_1^{\alpha^r_1} w_2^{\alpha^r_2} \cdots w_n^{\alpha^r_n+1} - \frac{k^r_{\mathrm{bw}}}{\beta^r_n + 1} w_1^{\beta^r_1} w_2^{\alpha^r_2} \cdots c_n^{\beta^r_n+1} \Big) \Big( \alpha^r_n - \beta^r_n \Big).$$

Notations: for $\mathbf{w} = (w_1, \ldots, w_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $r = 1, \ldots, R$:

$$\mathbf{w}^{\boldsymbol{\alpha}} := \prod_{k=1}^{n} w_k^{\alpha_k}, \ \boldsymbol{\alpha}^r := (\alpha^r_1, \ldots, \alpha^r_n), \ \boldsymbol{\beta}^r := (\beta^r_1, \ldots, \beta^r_n).$$

## Theorem

*Suppose that there exists* $\mathbf{w}_\infty \in \mathbb{R}^n_+$ *such that*

$$k^r_{fw}\mathbf{w}_\infty^{\boldsymbol{\alpha}^r} = k^r_{bw}\mathbf{w}_\infty^{\boldsymbol{\beta}^r} = \kappa_r \quad \text{for} \quad r = 1, \ldots, R, \qquad (*)$$

*then the above game is generalised potential.*

$(*)$: *detailed balanced condition*

## Proof.

$$\phi(\mathbf{w}) := \sum_{i=1}^{n} w_i \Big( \log\big(\frac{w_i}{w_{i\infty}}\big) - 1 \Big),$$

$$\Psi^*(\mathbf{w}, \boldsymbol{\mu}) = \sum_{r=1}^{R} \frac{\kappa_r}{2} \ell\Big(\frac{\mathbf{w}^{\boldsymbol{\alpha}^r}}{\mathbf{w}_\infty^{\boldsymbol{\alpha}^r}}, \frac{\mathbf{w}^{\boldsymbol{\beta}^r}}{\mathbf{w}_\infty^{\boldsymbol{\beta}^r}}\Big) \Big(\boldsymbol{\mu} \cdot (\boldsymbol{\alpha}^r - \boldsymbol{\beta}^r)\Big)^2 \quad \text{where}$$

$$\ell(a, b) = \begin{cases} \frac{a-b}{\log a - \log b} & \text{for} \quad a \neq b, \\ b \quad \text{for} \quad a = b. \end{cases}$$

# Origin of our generalization

classical gradient flow system: linear relation between rates and driving forces

- $\mathbb{R}^d$, standard inner product: $\dot{x}(t) = -\nabla f(x(t))$,
- $\mathbb{R}^d$, weighted inner product, $\langle x, y \rangle = x^T A y$: $\dot{x}(t) = -A^{-1} \nabla f(x(t))$.

generalized gradient flow systems: nonlinear relation between rates and driving forces

$$\dot{x}(t) = \mathbf{D}_\xi(\Psi^*(-\nabla \phi(x(t()))).$$

many applications in the modeling of materials (for examples, for plasticity and ferromagnetism) and in chemical reaction network.
*Our construction comes from chemical reaction network.*

# Future work

(i)(applications in machine learning) Decomposition of general games:
Balduzzi et. al. (ICML, 2018) & Letcher et.al. (JMLR, 2019)

$$\textit{general game} = \textit{exact potential game} + \textit{Hamiltonian game}$$

by decomposing the Hamiltonian into a sum of a symmetric and an anti-symmetric part using Helmholtz's decomposition.

on-going work: decomposition of games as in GENERIC-evolution (Ottinger, 2005)

$$\partial_t z = L(z)\frac{\delta E}{\delta z} + \partial_\xi \Psi^*(z, \frac{\delta S}{\delta z}),$$

where $L$: anti-symmetric operator, $E, S$: energy and entropy functional. GENERIC is both geometrical and physical/thermodynamical meaningful fulfilling the laws of thermodynamics for a closed system.

# Future work

(ii) (applying in robotics) optimization under uncertainty

Picheny et.al. (JGB, 2018): Bayesian optimization for black-box function.
on-going work: functional Bayesian optimization.

(iii) Game theory in chemical reaction networks: explore more connections between game theory and chemical reaction network.

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THANK YOU FOR YOUR ATTENTION!