Optimal assignments with supervisions

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January 24^{th} 2019

BASIC DEFINITIONS AND CONCEPTS

Tropical linear algebra

- Consider real numbers $\mathbb{R} \cup \{-\infty\}$ equipped with

$$a \odot b = a + b, \ \ a \oplus b := \max(a, b).$$

- Semifield with $\mathbf{0} = -\infty$, $\mathbf{1} = 0$.
  I.e. $a^{-1} = -a$ and $\nexists \ominus a$.
- Applies to matrices and vectors entry-wise:

$$(A \oplus B)_{i,j} := (A_{i,j} \oplus B_{i,j})$$

$$(A \odot B)_{i,j} := \bigoplus_k A_{i,k} \odot B_{kj}$$

Jacobi identity

Correspondence : $\mathbf{I}, \mathbf{J}$ minor of $\mathbf{A}^{-1}$ to $\mathbf{J^c}, \mathbf{I^c}$ minor of $\mathbf{A}$.

Theorem (the classical identity)

For $A \in \mathrm{GL}_n(\mathbb{F})$, $I, J \subseteq [n]$ *s.t.* $|I| = |J| = k$

$$(DA^{-1}D)^{\wedge k}_{I,J} = (\det(A))^{-1} A^{\wedge n-k}_{J^c, I^c},$$

where $D_{i,i} = (-1)^i$ *and* $D_{i,j} = 0$ *for* $i \neq j$.

(for instance) S. M. Fallat and C. R. Johnson, Totally Nonnegative Matrices. Princeton press, 2011.

Jacobi identity

Theorem (the tropical identity)

Let $M \in \mathbb{R}_{\max}^{n\times n}$ *and* $I, J \subseteq [n]$ *s.t.* $|I| = |J| = k$.
Either:

$$[D(\det(M)^{-1}\mathrm{adj}(M))D]_{I,J}^{\wedge k} = \det(M)^{-1} M_{J^c,I^c}^{\wedge n-k}$$

Or:
There exist distinct bijections $\pi, \sigma \in S_{I,J}$ *such that*

$$[\mathrm{adj}(M)]_{I,J}^{\wedge k} = \bigodot_{i\in I} \mathrm{adj}(M)_{i,\pi(i)} = \bigodot_{i\in I} \mathrm{adj}(M)_{i,\sigma(i)}.$$

M. Akian, S. Gaubert and N, Tropical Compound Matrix Identities, LAA, 2018.

How did it form?

The tropical *determinant* is actually the *permanent* with respect to $\oplus, \odot$. That is

$$\mathrm{per}(A) = \bigoplus_{\pi \in S_n} \bigodot_{i \in [n]} A_{i,\pi(i)} = \max_{\pi \in S_n} \sum_{i \in [n]} A_{i,\pi(i)},$$

Graphically: the permutation of optimal weight in the graph of A, Combinatorially: the 'optimal assignment problem'.

Let π, τ be permutations of identical weight w.
* In supertropical $w(\pi) \oplus w(\tau)$ is sigular.
* In symmetrized $w(\pi) \oplus w(\tau)$ is singular if π and τ are permutations of opposite signs.

How did it form?

- 2013 - PhD (with L.Rowen) - Conjecture: Let $A^{\nabla} = \text{per}^{-1}(A)\text{adj}(A)$ (sort of inverse). Then (supertropically) coefficient-wise
  $\text{per}(A)f_{A^{\nabla}}(x) = x^n f_A(x^{-1}) \oplus$ '*singular polynomial*'.

  That is, $\oplus \mathbf{A}^{\nabla}_{\mathbf{I},\mathbf{I}}$ corresponds to $\oplus \mathbf{A}_{\mathbf{I}^c,\mathbf{I}^c}$.
  [Y.Shitov 'On the Char. Polynomial of a Supertropical Adjoint Matrix', LAA.]

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- 2015 - Postdoc (with M.Akian and S.Gaubert) - (symmetrized) Tropical Jacobi:
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  So, **entry-wise**, for **every I**, **J**, and **including signs**.

- 2016-2018 (with McCaig and Sergeev) - Graph theory version:
  Every optimal $(1,k)$-regular multigraph of M w.r.t. I, J
  either: corresponds to an optimal bijection w.r.t. I^c, J^c,
  or: there exists another optimal $(1,k)$-regular w.r.t. I, J.
  [That is, combinatorially, without signs, which led to the application.]

Definitions: digraphs

- A weighted digraph G is a pair (V_G, E_G) where
  - V_G is **set of nodes** and
  - $E_G \subseteq V_G \times V_G$ is **set of directed edges** on $|V_G|$ nodes (allowing loops and multiple edges).
  - **Weight:** $w(i,j)$ for each (i,j).
- A bipartite graph is a triple $(V_{H,1}, V_{H,2}, E_H)$ s.t.

  $i \in V_{H,1} \Leftrightarrow j \in V_{H,2}$ for every $(i,j) \in E_H$, **weighted:** $w(i,j)$ for each (i,j).

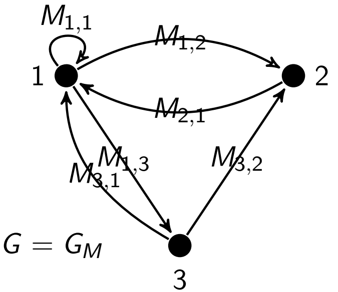
Associated digraphs

- **Matrix** $M \in \mathbb{R}_{\max}^{n \times n} \longrightarrow$ **weighted digraph** $G_M = (V, E)$, where $V = [n]$ and $E = \{(i,j) \colon M_{i,j} \neq \mathbf{0}\}$, and weight $w(i,j) = M_{i,j}$.
- **Weighted digraph** $G = ([n], E, w) \longrightarrow$ **matrix** M_G,

$$\text{where} \qquad (M_G)_{i,j} = \begin{cases} w(i,j) & ;\ \text{if } (i,j) \in E, \\ \mathbf{0} & ;\ \text{otherwise.} \end{cases}$$

Digraphs and matrices

$$M = M_G = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & \mathbf{0} & \mathbf{0} \\ M_{3,1} & M_{3,2} & \mathbf{0} \end{pmatrix} \longleftrightarrow$$



Associated bipartite graphs

- **Matrix** $M \in \mathbb{R}_{\max}^{m\times n}$ $\longrightarrow$**bipartite graph** $G_M = (V_{H_1}, V_{H_2}, E_H)$, $|V_{H_1}| = m$, $|V_{H_2}| = n$, and $E_H = \{(i,j)\colon M_{i,j} \neq -\infty\}$, weight $w(i,j) = M_{i,j}$.

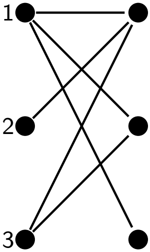
- **Bipartite graph** $G = (V_{H_1}, V_{H_2}, E_H)$ $\longrightarrow$**matrix** $M_G \in \mathbb{R}_{\max}^{m\times n}$ $|V_{H_1}| = m$, $|V_{H_2}| = n$

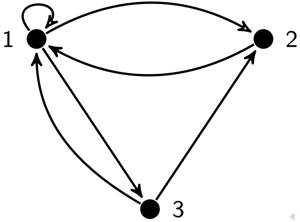
  where $$(M_G)_{i,j} = \begin{cases} w(i,j) & ; \text{ if } (i,j) \in E_H, \\ \mathbf{0} & ; \text{ otherwise.} \end{cases}$$

- **Digraph**$DG = ([n], E_D)$ $\longleftrightarrow$**bipartite graph**$BG = ([2n], E_B)$, s.t. $(i, j+n) \in E_B$ for every $(i,j) \in E_D$.

Bipartite graphs and matrices

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & \mathbf{0} & \mathbf{0} \\ M_{3,1} & M_{3,2} & \mathbf{0} \end{pmatrix} \longleftrightarrow$$





Definitions: assignment problems

- Let S_n denote the set of permutations on $[n]$, and $S_{I,J}$ denote the set of bijections from $I \subseteq [n]$ to $J \subseteq [n]$ (that is, $|I| = |J|$).
- For $M \in \mathbb{R}_{\max}^{n\times n}$ tropical permanent is defined by

$$\operatorname{per}(M) = \max_{\pi\in S_n} \sum_{i\in[n]} M_{i,\pi(i)} = \bigoplus_{\pi\in S_n} \bigodot_{i\in[n]} M_{i,\pi(i)}.$$

- A permutation π of maximal weight in $\operatorname{per}(M)$ is an optimal permutation in M or G_M. That is,

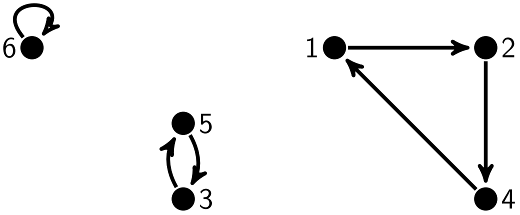
$$\operatorname{per}(M) = \bigodot_{i\in[n]} M_{i,\pi(i)} = \sum_{i\in[n]} w(i,\pi(i)).$$

- This is identical to the set of **optimal assignments**, i.e., optimal solutions to the assignment problem in the bipartite graph associated with M.

Permutation subgraphs

- A non-$\mathbf{0}$ tropical "summand" $w(\pi) = \bigodot_{i\in[n]} M_{i,\pi(i)}$ in per$M$, or in M $\leftrightarrow$ **permutation-subgraph** of G_M

  with $V(E_\pi) = [n]$, $E_\pi = \{(i, \pi(i))\ \forall i \in [n]\}$.

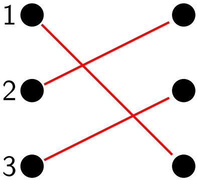


(1 2 4)(5 3)(6)

(and the same for path, cycle, bijection,...)

Assignment subgraphs

- A non-**0** tropical "summand" $w(\pi) = \bigodot_{i\in[n]} M_{i,\pi(i)}$ in per$M$ $\leftrightarrow$ **assignment subgraph** with $V(E_\pi) = [n] + [n]$, $E_\pi = \{(i, \pi(i))\ \forall i \in [n]\}$.



(2 1 3)

(and the same for path, cycle, bijection,...)

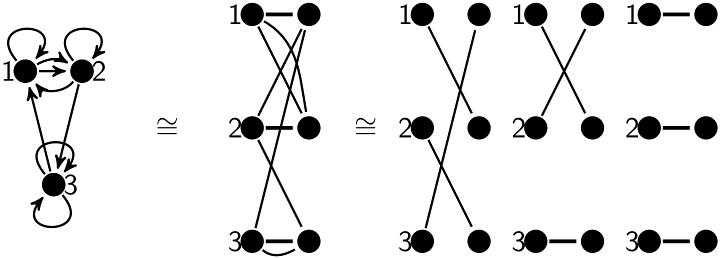
k-regular graphs

- A graph or digraph $G = (V, E)$ is k-regular if
  $\forall v \in V$: $\deg(v) = k$ (if G is a graph)
  $\forall v \in V$: $\deg^+(v) = \deg^-(v) = k$ (if G is a digraph).
- **Observation:** Let $G = ([n], E)$ be a k-regular digraph, then

$$E = \biguplus_{i \in [k]} E_{\rho_i}, \ \ \rho_i \in S_n$$

  i.e., a disjoint union of edge sets of k permutation-subgraphs $G_i = ([n], E_{\rho_i})$ for some ρ_i, for $i \in [k]$.
  [Hall's Marriage Thm and Z.Izhakian and L.Rowen, Supertropical matrix algebra.]
- So $G = (([n], \biguplus_{i \in [k]} E_{\rho_i})$.

Hall's Marriage Theorem



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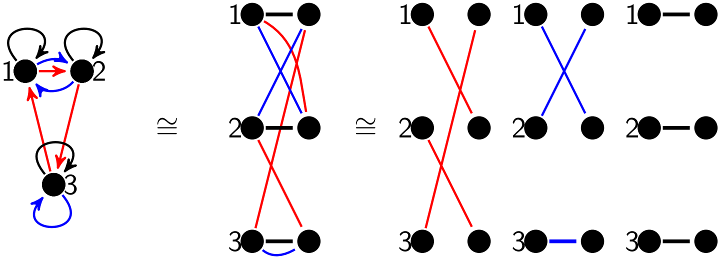


$(1, k)$-regular graphs

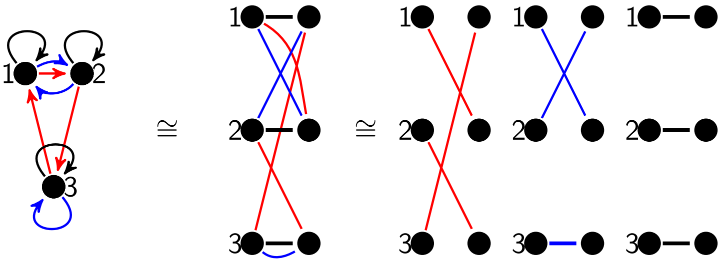
- Let G be k-regular (with $\rho_1, \ldots, \rho_k$). We say G is $(1, k)$-regular w.r.t. I, J with $|I| = |J| = k$ if there exist $e_i \in E_{\rho_i}$ $\forall i \in [k]$ s.t. $s(e_i) \in I$, $t(e_i) \in J$ and

$$\left(V(E_\pi), E_\pi = \{e_1, \ldots, e_k\}\right)$$

  is a bijection-subgraph.
- We denote

$$G = \left([n], \biguplus_{i \in [k]} E_{\rho_i}, \pi\right).$$

Example: (1,3)-regular graph



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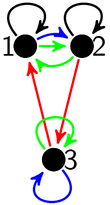

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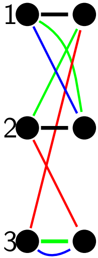

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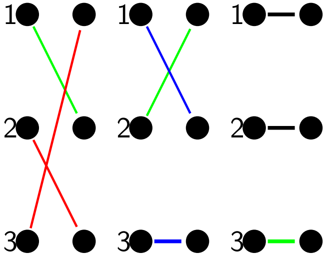


Tropical adjugate

- Denote by $M^{\wedge k} \in \mathbb{R}_{\max}^{\binom{n}{k} \times \binom{n}{k}}$ the tropical k^{th} compound matrix of M defined by

$$M^{\wedge k}_{I,J} = \bigoplus_{\sigma \in S_{I,J}} \bigodot_{i \in I} M_{i,\sigma(i)} = \max_{\sigma \in S_{I,J}} \sum_{i \in I} M_{i,\sigma(i)}$$

$\forall I, J \subseteq [n] : \; |I| = |J| = k, \; I, J$ ordered lexicographically.

- In particular, $M^{\wedge 1} = M, \; M^{\wedge 0} = \mathbf{1}$ and $\mathrm{per}(M) = M^{\wedge n}$ is the tropical permanent of M.

- $\mathrm{adj}(M)_{i,j} = M^{\wedge n-1}_{\{j\}^c, \{i\}^c}$
is the $(i\ j)$ entry of the tropical adjugate of M.

Optimal $(1,k)$-regular multigraphs

- We say that $\left([n], \biguplus_{i\in[k]} E_{\rho_i}, \sigma\right)$ is an **optimal** $(1,k)$-regular multigraph of G w.r.t. I, J if

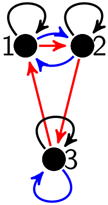
$$\left(\sum_{i\in[k]} w(\rho_i)\right) - w(\sigma) \geq \left(\sum_{i\in[k]} w(\rho'_i)\right) - w(\sigma'),$$

  for every $(1,k)$-regular multigraph $\left([n], \biguplus_{i\in[k]} E_{\rho'_i}, \sigma'\right)$ of G.
- **Equivalently**

$$\left(\operatorname{adj}(M_G)\right)^{\wedge k}_{J,I} = \bigodot_{i\in I} \left(\operatorname{adj}(M_G)\right)_{\sigma(i),i}\ , \text{ where}$$

$$\left(\operatorname{adj}(M_G)\right)_{\sigma(i),i} = \bigodot_{j\in\{i\}^c} (M_G)_{j,\rho_i(j)}.$$

Example: (1,3)-regular graph



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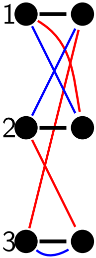

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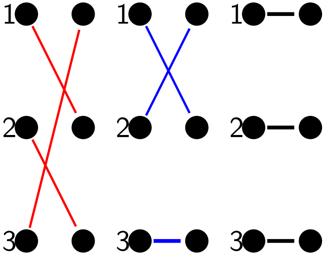

$$\left(\sum_{i\in[3]} w(\rho_i)\right)$$

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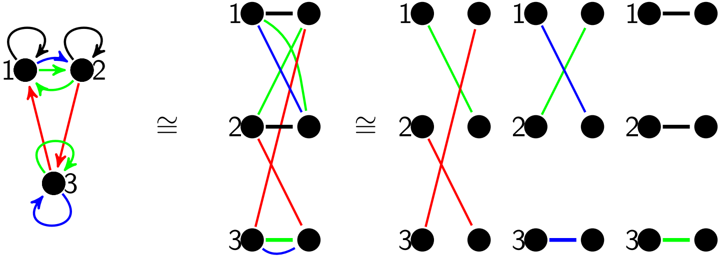
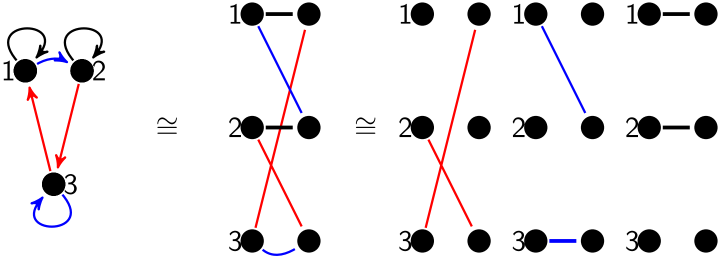


$$\left(\sum_{i\in[3]} w(\rho_i)\right) \ , \ w(\sigma)$$

Example: (1,3)-regular graph



$$\left(\sum_{i \in [3]} w(\rho_i) \right) - w(\sigma) = \left(\mathrm{adj}(M_G)\right)^{\wedge k}_{J,I}$$

OPTIMAL ASSIGNMENTS WITH SUPERVISIONS

Assignments with supervisions

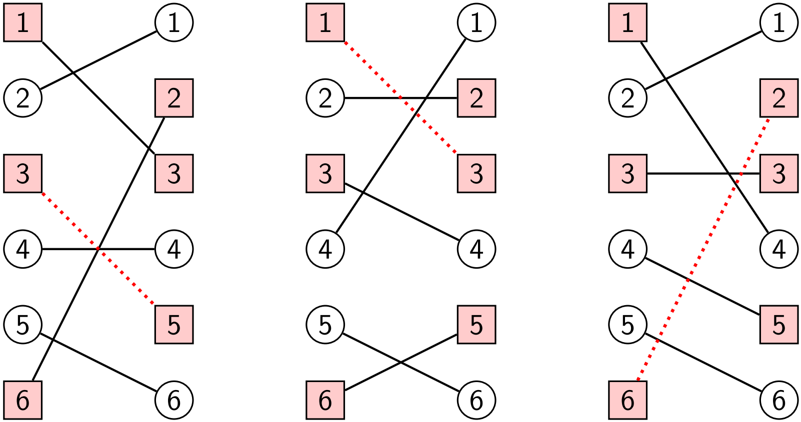
- Supervisions: Let
  - $M \in \mathbb{R}_{\max}^{n \times n}$
  - $\rho_t \in S_n$ for $t \in [k]$ be k assignments,
  - $(i_t, j_t) \in I \times J$ be k edges s.t. $\sigma(i_t) = j_t$ for $\sigma \in S_{I,J}$.

  σ defines supervisions on $\{\rho_t : t \in [k]\}$ if $\rho_t(i_t) = j_t \ \forall t$.
- The base value of these assignments with supervisions is

$$\sum_{t \in [k]} \left(w(\rho_t, M) - M_{i_t, \sigma(i_t)} \right) = \sum_{t=1}^{k} \sum_{i \neq i_t} M_{i, \rho_t(i)}.$$

- This is also the **weight of** $(1, k)$**-regular multigraph** $([n], \biguplus_{t \in [k]} E_{\rho_t}, \sigma)$.

Assignments with supervisions of people $\{1, 3, 6\}$ on tasks $\{2, 3, 5\}$



Key observation

- The optimal base value of k assignments with supervisions I on J is
$$\bigoplus_{\sigma \in S_{J,I}} w(\sigma, \mathrm{adj}(M)_{J,I}) = [\mathrm{adj}(M)]^{\wedge k}_{J,I}$$
- It is also the the weight of an optimal $(1, k)$-regular multigraph w.r.t. I and J.

Example

- Let

$$M = \begin{pmatrix} 0 & 1 & -2 & -4 \\ -3 & 0 & 5 & 2 \\ -5 & 4 & 0 & 6 \\ -1 & -6 & 3 & 0 \end{pmatrix}, \text{ then } \operatorname{adj}(M) = \begin{pmatrix} 9 & 10 & 6^\bullet & 12 \\ 10 & 9 & 5^\bullet & 11 \\ 5 & 6 & 2 & 6^\bullet \\ 8 & 9 & 5 & 9 \end{pmatrix}.$$

- **Goal**: Find optimal assignments with supervisions of $I = \{2,4\}$ on $J = \{1,2\}$.
- The **maximum base value** is given by

$$\operatorname{adj}(M)^{\wedge 2}_{J,I} = \operatorname{per}\begin{pmatrix} 10 & 12 \\ 9 & 11 \end{pmatrix} = 21^\bullet.$$

- The **optimal bijections (supervisions)** are $\sigma_1 = (2 \to 1)(4 \to 2)$ and $\sigma_2 = (2 \to 2)(4 \to 1)$.

Example: the end of solution

- We found that $\sigma_1 : (2 \to 1)(4 \to 2)$ is optimal.
- Supervision $2 \to 1$ corresponds to

$$M_{\{1,3,4\},\{2,3,4\}} = \begin{pmatrix} \mathbf{1} & -2 & -4 \\ 4 & 0 & \mathbf{6} \\ -6 & \mathbf{3} & 0 \end{pmatrix}.$$

$$\beta_1 = (1 \to 2)(3 \to 4)(4 \to 3) \in S_{\{1,3,4\},\{2,3,4\}},$$

$$\rho_1 = (1 \to 2)(\mathbf{2} \to \mathbf{1})(3 \to 4)(4 \to 3) \in S_4.$$

- For supervision $4 \to 2$, we similarly obtain:
  $\beta_2 = (1 \to 1)(2 \to 3)(3 \to 4) \in S_{\{1,2,3\},\{1,3,4\}}$,
  $\rho_2 = (1 \to 1)(2 \to 3)(3 \to 4)(\mathbf{4} \to \mathbf{2}) \in S_4$.
- **Optimal $(1, k)$-regular multigraph:**
  $F = (E_{\rho_1} \uplus E_{\rho_2}, \sigma_1)$.

TROPICAL JACOBI IDENTITY IN GRAPHS

(non-symmetrized) Tropical Jacobi identity

Theorem (Tropical Jacobi identity)

Let $M \in \mathbb{R}_{\max}^{n\times n}$ and $I, J \subseteq [n]$ such that $|I| = |J| = k$. Then:

1. $[\mathrm{per}(M)^{-1}\mathrm{adj}(M)]^{\wedge k}_{I,J} = \mathrm{per}(M)^{-1} M^{\wedge n-k}_{J^c,I^c}$ **OR**
2. *There exist distinct bijections $\pi, \sigma \in S_{I,J}$ such that*

$$[\mathrm{adj}(M)]^{\wedge k}_{I,J} = \sum_{i\in I} \mathrm{adj}(M)_{i,\pi(i)} = \sum_{i\in I} \mathrm{adj}(M)_{i,\sigma(i)}.$$

[M. Akian, S. Gaubert and N, Tropical compound matrix identities, LAA.]

Tropical adjugate and optimal multigraphs

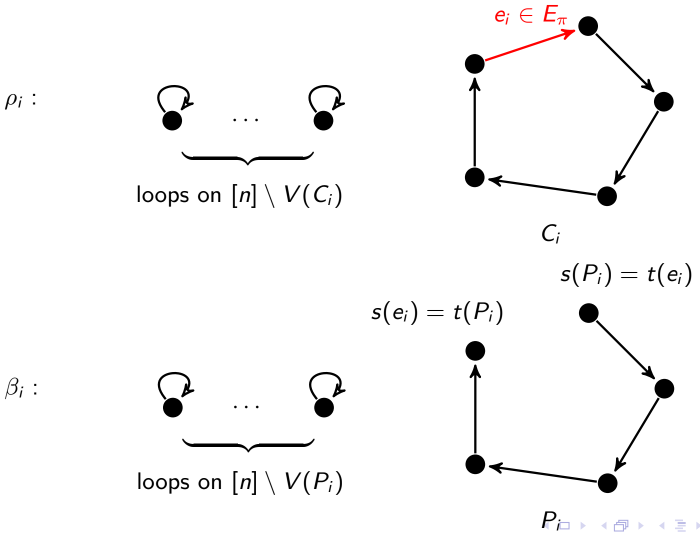
- $(\mathrm{adj}M)^{\wedge k}_{J,I} =$
  the weight of an optimal $(1,k)$**-regular multigraph** $F = \left([n], \biguplus_{i\in[k]} E_{\rho_i}, \pi\right)$ w.r.t. $I, J \subseteq [n]$.

- We will assume that $M_{i,i} = \mathbf{1}$ *and* $\mathrm{Id} \in S_n$ is an optimal assignment in M. That is, $\mathrm{per}(M) = \bigodot_{i\in[n]} M_{i,i} = \mathbf{1}$.

  Indeed, this normalization $M \mapsto PM$ process is invertible, so by Binet-Cauchy and classical Jacobi, if tropical Jacobi holds for PM, it holds for M.

- This means $\mathrm{Id} \in S_k$ is an optimal assignment of weight $\mathbf{1}$ in M for every k, and in particular, loops are 'equally or more optimal' than every cycle.

Case of unicycle permutations ρ_i



Tropical Jacobi identity in multigraphs

Theorem

- *Let* Id *be an optimal permutation in* $G = ([n], E)$.
- *Let* $F = \left([n], \biguplus_{i\in[k]} E_{\rho_i}, \pi\right)$ *be an optimal* $(1, k)$*-regular multigraph of* G *with respect to* $I, J \subseteq [n]$.

EITHER:

$w(F) = w(\sigma)$ *where* $\sigma \in S_{I^c,J^c}$ **is an optimal bijection,**

OR:

There exists $\tilde{\pi} \in S_{I,J}$ *and* $\tau_i \in S_n$ *s.t.* $F' = \left([n], \biguplus_{i\in[k]} E_{\tau_i}, \tilde{\pi}\right) \neq F$ **is also an optimal** $(1, k)$**-regular multigraph** *with respect to* I, J.

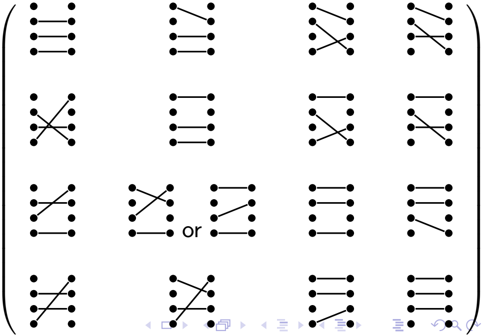
Example

- Let

$$A = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -6 & 0 & -2 & -1 \\ -3 & -4 & 0 & -3 \\ -2 & -7 & 0 & 0 \end{pmatrix}.$$

- Then

$$\operatorname{adj}(A) = \begin{pmatrix} 0 & -1 & -2 & -2 \\ -3 & 0 & -1 & -1 \\ -3 & -4^{\bullet} & 0 & -3 \\ -2 & -3 & 0 & 0 \end{pmatrix} \leftrightarrow$$



The first case

- **Case 1: All paths P_i for $i \in [k]$ are pairwise disjoint.**
- Under this condition, we take
  $\sigma =$ composition $P_1 \circ \ldots \circ P_k$ with disjoint loops. That is:
  (a) All sources and targets of P_i are disjoint,
  (b) Sources and targets are disjoint to all intermediate nodes,
  (c) All intermediate nodes of P_i are disjoint.

The first case

$$
\begin{array}{ccccccccccccc}
(1,k)- & & & & S_n & & \mathrm{Id}^{k-1} & & \begin{array}{c} S_{I^c,J^c} \\ \Cup \\ \pi' \end{array} & & \begin{array}{c} S_{I,J} \\ \Cup \\ \pi \end{array} & & \\
\text{regular} & & & & \Cup & & \| & & \| & & \| & & \\
\begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix} & \begin{matrix} = \\ = \\ \vdots \\ = \end{matrix} & \begin{pmatrix} \rho_1|_{\overline{s(e_1)}^c} \\ \rho_2|_{\overline{s(e_2)}^c} \\ \vdots \\ \rho_k|_{\overline{s(e_k)}^c} \end{pmatrix} & \begin{matrix} \circ \\ \circ \\ \vdots \\ \circ \end{matrix} & \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{pmatrix} & \begin{matrix} = \\ = \\ \vdots \\ = \end{matrix} & \begin{pmatrix} \text{loops} \\ \text{loops} \\ \vdots \\ \text{loops} \end{pmatrix} & \begin{matrix} \circ \\ \circ \\ \vdots \\ \circ \end{matrix} & \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{pmatrix} & \begin{matrix} \circ \\ \circ \\ \vdots \\ \circ \end{matrix} & \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{pmatrix} & & \begin{matrix} \tau_1 = \mathrm{Id} \\ \vdots \\ \tau_{k-1} = \mathrm{Id} \\ \tau_k = \pi' \circ \pi \end{matrix} \\
& & & & \circ & & & & \circ & & & & \\
& & & & \text{loops} & & & & \text{loops} & & & &
\end{array}
$$

Figure: Case (1): Optimal $(1,k)$-regular multigraph F corresponds to an

Example: Case 1

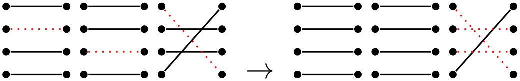
$$A = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -6 & 0 & -2 & -1 \\ -3 & -4 & 0 & -3 \\ -2 & -7 & 0 & 0 \end{pmatrix}, \quad \mathrm{adj}(A) = \begin{pmatrix} 0 & -1 & -2 & -2 \\ -3 & 0 & -1 & -1 \\ -3 & -4^{\bullet} & 0 & -3 \\ -2 & -3 & 0 & 0 \end{pmatrix}$$

Case 1 in the theorem:

$$\mathrm{adj}(A)^{\wedge 3}_{\{1\}^c,\{4\}^c} = -2 = A_{4,1} = A^{\wedge 1}_{\{4\},\{1\}},$$

Example: Case 1

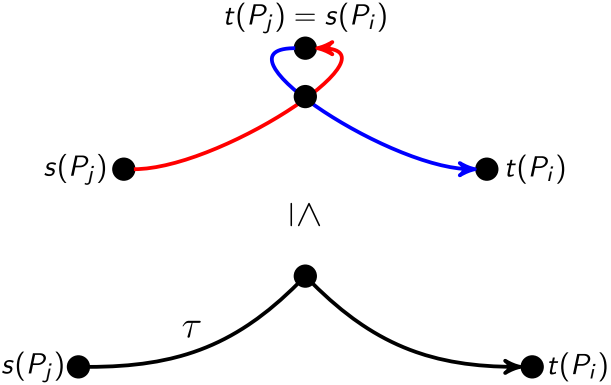
- **Left:** $\mathrm{adj}(A)^{\wedge 3}_{\{2,3,4\},\{1,2,3\}}$ is the weight of F (an optimal $(1,k)$-regular multigraph).
- **Right:** σ is the (optimal) bijection: $J^c = \{4\} \to I^c = \{1\}$. Joined with loops and the supervision edges, it makes a permutation.



$$(2)\ (3)\ 4 \to 1$$

Violation of a)

Case 2a): There exists a source which is also a target.
In this case $\exists i, j \in [k] : t(P_j) = s(P_i)$.



composed with disjoint loops

Violation of a)

Construct $F' = (\biguplus_{i\in[k]} E_{\tau_i}, \pi')$ by:

- Replacing $\rho_i, \rho_j \longrightarrow (\tau_i = \tau), (\tau_j = \mathrm{Id})$,
- Keeping $\tau_\ell = \rho_\ell$ for all $\ell \neq i, j$,
- $\tilde{\pi}$ is formed from π by replacing $(t(P_j), s(P_j)),\ (t(P_i), s(P_i)) \longrightarrow (t(P_i), s(P_j)),\ (t(P_j), s(P_i))$.

Example: Case 2a

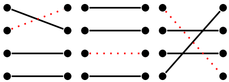
$$A = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -6 & 0 & -2 & -1 \\ -3 & -4 & 0 & -3 \\ -2 & -7 & 0 & 0 \end{pmatrix}, \quad \mathrm{adj}(A) = \begin{pmatrix} 0 & -1 & -2 & -2 \\ -3 & 0 & -1 & -1 \\ -3 & -4^{\bullet} & 0 & -3 \\ -2 & -3 & 0 & 0 \end{pmatrix}$$

Case 2a in the theorem For $I = \{1,2,3\}$ and $J = \{1,3,4\}$ we have

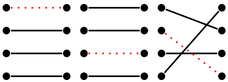
$$\mathrm{adj}(A)^{\wedge 3}_{J,I} = -3^{\bullet} > A^{\wedge 1}_{I^C,J^C} = A_{4,2} = -7.$$

Example: Case 2a

- **Left** is attained by two bijections in adj(A): $(3), 4 \to 1 \to 2$ and $(1)(3), 4 \to 2$.
- These bijections represent, in A, the following choices for 3 assignments with supervisions:



and



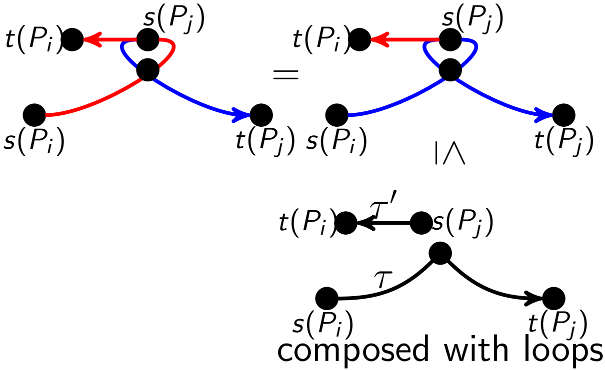
obtained by the same set of reorganized edges:

$$4 \to 1 \;\; 1 \to 2 \; and \; 4 \to 1 \to 2$$

Violation of b)

Case 2b: There exists an intermediate node which is also a source or a target.

Assume w.l.o.g. that Case 2a does not occur.



Violation of b)

Construct $F' = (\biguplus_{i\in[k]} E_{\tau_i}, \pi')$ by:

- Replacing $\rho_i, \rho_j \longrightarrow (\tau_i = \tau), (\tau_j = \tau')$,
- Keeping $\tau_\ell = \rho_\ell$ for all $\ell \neq i, j$,
- $\tilde{\pi}$ is formed from π by replacing $(t(P_j), s(P_j)), (t(P_i), s(P_i) \longrightarrow (t(P_i), s(P_j)), (t(P_j), s(P_i))$.

Example: Case 2b

$$A = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -6 & 0 & -2 & -1 \\ -3 & -4 & 0 & -3 \\ -2 & -7 & 0 & 0 \end{pmatrix}, \quad \mathrm{adj}(A) = \begin{pmatrix} 0 & -1 & -2 & -2 \\ -3 & 0 & -1 & -1 \\ -3 & -4^{\bullet} & 0 & -3 \\ -2 & -3 & 0 & 0 \end{pmatrix}$$

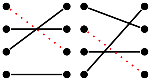
Case 2b in the theorem For $I = \{1,2\}$ and $J = \{3,4\}$ we have:

$$\mathrm{adj}(A)^{\wedge 2}_{J,I} = -6^{\bullet} = (\mathrm{adj}(A)_{3,1}\mathrm{adj}(A)_{4,2}) \oplus (\mathrm{adj}(A)_{3,2}\mathrm{adj}(A)_{4,1}),$$

$$A^{\wedge 2}_{I^C,J^C} = -6 = A_{3,2}A_{4,1}.$$

Example: Case 2b

- In this case $\text{adj}(A)^{\wedge 2}_{J,I}$ is attained twice AND equality holds in the tropical Jacobi identity.
- There are three sets of 2 assignments obtaining the optimal base value: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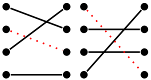
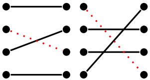

and


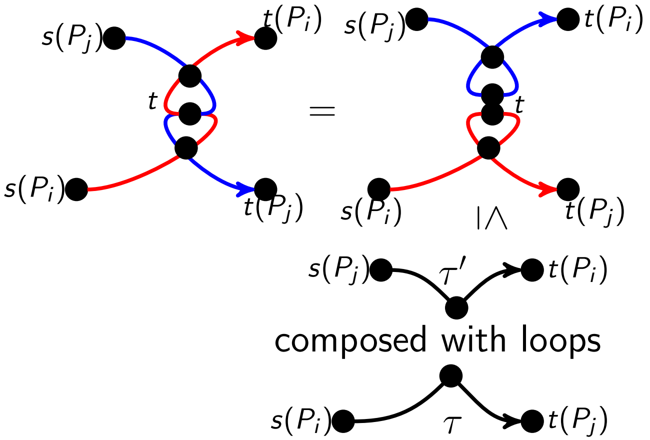
The first two are obtained by the same set of reorganized edges:

$$3 \to 1 \quad 4 \to 1 \to 2 \quad \textit{and} \quad 4 \to 1 \quad 3 \to 1 \to 2$$

The third is case1 - disjoint paths: $3 \to 2 \quad 4 \to 1$ obtaining $A^{\wedge 2}_{I^C, J^C}$.

Violation of c)

Case 2c): There exists an intermediate node common to two paths. Assume w.l.o.g. that Cases 2a,2b do not occur.



Violation of c)

Construct $F' = (\biguplus_{i\in[k]} E_{\tau_i}, \pi')$ by:

- Replacing $\rho_i, \rho_j \longrightarrow (\tau_i = \tau), (\tau_j = \tau')$,
- Keeping $\tau_\ell = \rho_\ell$ for all $\ell \neq i, j$,
- $\tilde{\pi}$ is formed from π by replacing $(t(P_j), s(P_j)), (t(P_i), s(P_i) \longrightarrow (t(P_i), s(P_j)), (t(P_j), s(P_i))$.

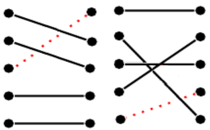
Example: Case 2C

$$(4)\ (5)\ 1 \to 2 \to 3 \quad , \quad (1)\ (3)\ 4 \to 2 \to 5$$

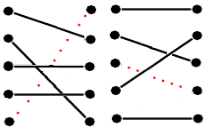
with the bijection $3 \to 1$, $5 \to 4$, becomes

$$(3)\ (4)\ 1 \to 2 \to 5 \quad , \quad (1)\ (5)\ 4 \to 2 \to 3$$

with the bijection $5 \to 1$, $3 \to 4$:



and



Supervised assignment optimization

Monday Tuesday Wednesday Thursday

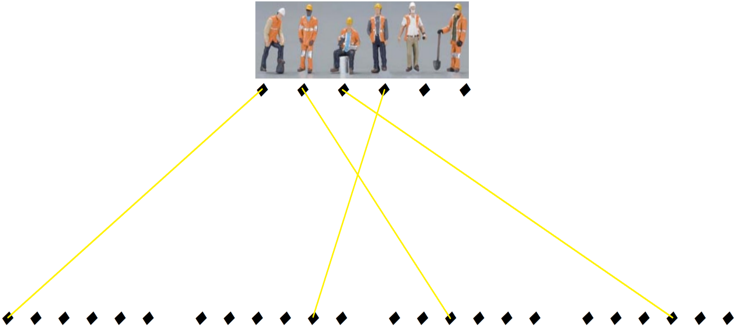
1. Work schedule 2. Lunch 3. Tips
4. Carpool 5. Inventory 6. Leftovers

Supervised assignment optimization

Monday Tuesday Wednesday Thursday

1. Work schedule (Monday) 2. Lunch 3. Tips (Wednesday)
4. Carpool 5. Inventory (Tuesday) 6. Leftovers (Thursday)

Supervised assignment optimization



Monday Tuesday Wednesday Thursday

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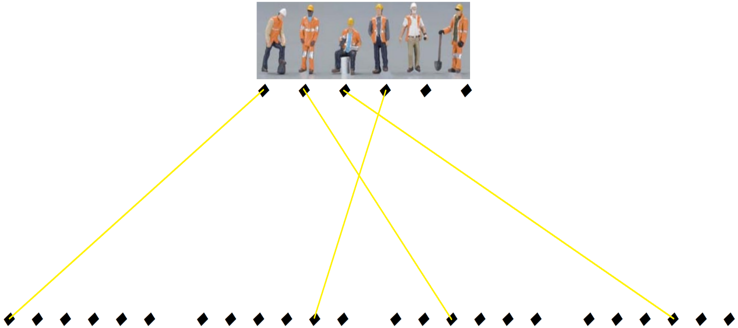
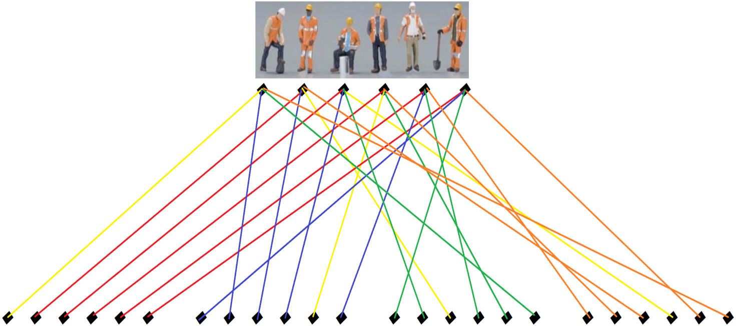
Supervised assignment optimization



$$(1)\ 2 \to 3 \to 4 \to 5 \in S_{\{1,2,3,4\},\{1,3,4,5\}}$$

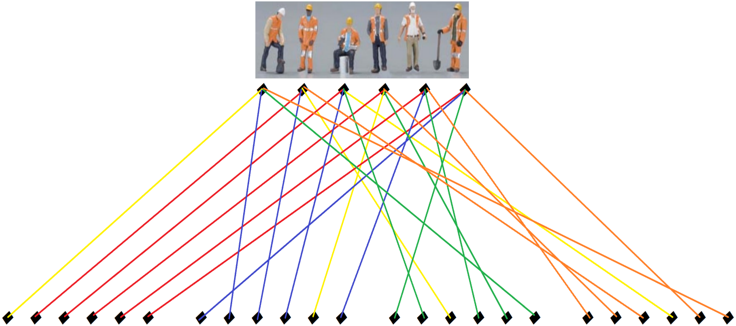
Supervised assignment optimization



$(1)(2)(3)(4)(5)(6) \quad (1\ 2\ 3\ 4\ 5\ 6)) \quad (2\ 3)(4\ 5)(1\ 6) \quad (2\ 3\ 4)(1\ 6\ 5)$

$(1)\ 2 \to 3 \to 4 \to 5 \in S_{\{1,2,3,4\},\{1,3,4,5\}}$

Supervised assignment optimization

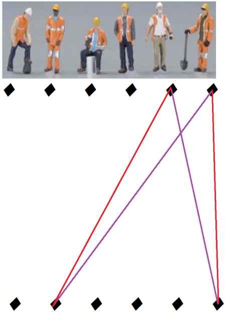


$(1)(2)(3)(4)(5)(6) \quad (1\ 2\ 3\ 4\ 5\ 6)) \quad (2\ 3)(4\ 5)(1\ 6) \quad (2\ 3\ 4)(1\ 6\ 5)$

$$(1)\ 2 \to 3 \to 4 \to 5 \in S_{\{1,2,3,4\},\{1,3,4,5\}}$$

$$(6)\ 5 \to 2 \quad \textit{or} \quad 5 \to 6 \to 2 \in S_{\{5,6\},\{2,6\}}$$

Supervised assignment optimization



$$(6)\ 5 \to 2 \quad or \quad 5 \to 6 \to 2 \in S_{\{5,6\},\{2,6\}}$$

THANK YOU!